Problemy Matematyczne
15 (1997), 47–55

Construction of some Finite Classes of Normal Subgroups of Hecke Groups

İsmail Naci Cangül, Osman Bizim

1. Introduction Hecke groups $H(\lambda q)$ are, basically, the discrete groups of PSL$(2, R)$ which are isomrphic to the free product of two cyclic groups of orders 2 and q, where $q \geq 3$. Therefore as a Fuchsian group of the first kind they have the signature $(0; 2, q, \infty)$ (here ∞ means that the product of two elliptic generators is a parabolic generator). $H(\lambda q)$ is generated by R and S, where $R(z) = -1/z$, $S(z) = 1/(z + \lambda q)$ and $\lambda q = 2\cos(\pi/q)$. We let $T = R \cdot S$.

For $q = 3$ we obtain the well known modular group $\Gamma = H(\lambda_3)$. Therefore the other Hecke groups can be thought of as a generalization of the modular group. At this point a problem arises: Can we generalize the properties of Γ to other $H(\lambda q)$? The answer to the first question is yes and no, that is, some of them can be generalized in some cases. Sometimes it is possible to generalize some properties to $H(\lambda q)$. However, some results generalize to $H(\lambda q)$ with even q as well.

In this work we obtain some finite classes of normal subgroups of Hecke groups as a generalization of some results of M. Newman [3] on the modular group. First we construct this class for the most important three Hecke groups $H\left(\sqrt{2}\right)$, $H\left(\sqrt{3}\right)$ and $H(\lambda_5)$ obtained for $q = 4, 6$ and 5, respectively. Then we obtain the generalization for all $H(\lambda q)$. Some of these sugroups have been found, in different ways, in the Ph. D. Thesis of the author [1].

In the last part of this work we calculate the total number $\eta(q)$ of these normal subgroups for each q by means of the number of divisors function. We shall see that $\eta(q)$ is either equal to $2\sigma(q)$ or $3\sigma(q)$, where $\sigma(q)$ denotes the number of divisors of q.

2. Normal subgroups

2.1. q =4 For $q = 4$, $\lambda_4 = \sqrt{2}$ and the corresponding group is denoted by $H\left(\sqrt{2}\right)$. We first give two preliminary lemmas to obtain the main result:

Lemma 1 *Let, for* $1 \leq k \leq 3$,

$$S_k\left(\sqrt{2}\right) = \langle R, S | R^2 = S^4 = I, (RS)^k = (SR)^k \rangle. \tag{1}$$

Then $S_k\left(\sqrt{2}\right)$ *is a finite group of order* μ_k *and for* $T = RS$, *the element* T^k *is central and has order* e_k, *where*

k	1	2	3
e_k	4	4	8
μ_k	8	32	192

(2)

Proof. First we note that since $R^2 = S^4 = I$ and $T = RS$, the relation $(RS)^k = (SR)^k$ is equivalent to the relation

$$RT^k = T^k R \quad \text{and} \quad ST^k = T^k S \tag{3}$$

and therefore T^k is a central element in $S_k\left(\sqrt{2}\right)$. Now if e_k denotes the order of T^k in $S_k\left(\sqrt{2}\right)$, then by [2, §6,7]

$$e_k = \frac{2 \cdot 2 \cdot 4}{t} = \frac{8}{4-k} \tag{4}$$

since t is defined by

$$t = 2 \cdot 4 \cdot k \cdot \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{k} - 1\right) = 8 - 2k. \tag{5}$$

Also

$$\left|S_k\left(\sqrt{2}\right)\right| = \mu_k = |(2,4,k)| = \frac{8k}{4-k} \cdot \frac{8}{4-k} = \frac{64}{(4-k)^2} \tag{6}$$

for $2 \leq k \leq 3$. thus for $k \neq 1$

$$\mu_k = ke_k^2 \tag{7}$$

and for $k = 1$, $e_1 = 4$ and $\mu_1 = 8$, since the group $S_1\left(\sqrt{2}\right)$ is just abelianisation of $H\left(\sqrt{2}\right)$.

Lemma 2 *For $1 \leq k \leq 3$, let $S_{k,l}\left(\sqrt{2}\right)$ be the group given by*

$$S_1\left(\sqrt{2}\right) = \langle R, S | R^2 = S^4 = T^{kl} = I, \quad RT^k = T^k R\rangle, \tag{8}$$

where $l|e_k$. Then $S_{k,l}\left(\sqrt{2}\right)$ is of order $l\mu_k/e_k$.

Proof. Let Δ_l be the normal closure in $S_{k,l}\left(\sqrt{2}\right)$ of T^{kl}. Then

$$S_{k,l}\left(\sqrt{2}\right) \cong S_k\left(\sqrt{2}\right)/\Delta_l \tag{9}$$

Since T^k is in the center of $S_k\left(\sqrt{2}\right)$, Δ_1 is the cyclic group of order e_k/l generated by T^{kl}. Hence

$$\left|S_{k,l}\left(\sqrt{2}\right)\right| \equiv \left|S_k\left(\sqrt{2}\right)\right| / |\Delta_l| = \frac{l\mu_k}{e_k}. \tag{10}$$

Theorem 1 *Let*

$$T_{k,l}\left(\sqrt{2}\right) = \Delta\left(T^{kl}, RT^kRT^{-k}\right), \tag{11}$$

be the normal closure in $H\left(\sqrt{2}\right)$ of T^{kl} and RT^kRT^{-k}. Then

$$\left|H\left(\sqrt{2}\right) : T_{k,l}\left(\sqrt{2}\right)\right| \mu = l\mu_k/e_k < \infty, \tag{12}$$

Proof. Since

$$\left|H\left(\sqrt{2}\right)/T_{k,l}\left(\sqrt{2}\right)\right| \cong S_{k,l}\left(\sqrt{2}\right) \tag{13}$$

the results follows.

One can see that there are exactly 10 normal subgroups $T_{k,l}\left(\sqrt{2}\right)$'s. Their indices, levels, parabolic class and geni for all possible values of k and l are listed below:

$$\begin{array}{c|cccccccccc} k & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ l & 1 & 2 & 4 & 1 & 2 & 4 & 1 & 2 & 4 & 8 \\ \mu & 2 & 4 & 8 & 8 & 16 & 32 & 24 & 48 & 96 & 192 \end{array} \qquad (14)$$

2.2. q = 6 Let us consider now the case of $H\left(\sqrt{3}\right)$. As all methods used in this case are similar to the case of $H\left(\sqrt{2}\right)$, then we will omit the proofs.

Lemma 3 *Suppose that $k = 1, 2$. Let $S_k\left(\sqrt{3}\right)$ be the group given by*

$$S_k\left(\sqrt{3}\right) = \langle R, S | R^2 = S^6 = I, (RS)^k = (SR)^k \rangle. \qquad (15)$$

Then $S_k\left(\sqrt{3}\right)$ is a finite group of order μ_k and the central element T^k has order e_k, where $T = RS$ and

$$\begin{array}{c|cc} k & 1 & 2 \\ \hline e_k & 6 & 6 \\ \hline \mu_k & 12 & 72 \end{array} \qquad (16)$$

Lemma 4 *For $k = 1, 2$ let $S_{k,l}\left(\sqrt{3}\right)$ be the group given by*

$$S_{k,l}\left(\sqrt{3}\right) = \langle R, S | R^2 = S^6 = T^{kl} = I, RT^k = T^k R \rangle. \qquad (17)$$

Then $S_{k,l}\left(\sqrt{3}\right)$ is of order $l\mu_k / e_k$.

Theorem 2 *Let*

$$T_{k,l}\left(\sqrt{3}\right) = \Delta\left(T^{kl}, RT^k RT^{-k}\right). \tag{18}$$

Then $T_{k,l}\left(\sqrt{3}\right)$ is a normal subgroup of $H\left(\sqrt{3}\right)$ with index

$$\left|H\left(\sqrt{3}\right) : T_{k,l}\left(\sqrt{3}\right)\right| = \mu = l\mu_k/e_k < \infty \tag{19}$$

and level

$$n = kl \tag{20}$$

There are exactly eight $T_{k,l}\left(\sqrt{3}\right)$'s. The information about them is listed below:

k	1	1	1	1	2	2	2	2
l	1	2	3	6	1	2	3	6
μ	2	4	6	12	12	24	36	72

(21)

2.3. q = 5 Let us finally consider $H(\lambda_5)$ case. All the methods are also very similar to those of the case of $H\left(\sqrt{2}\right)$, then we will omit the proofs.

Lemma 5 *For $k = 1, 2, 3$ let*

$$S_k(\lambda_5) = \langle R, S | R^2 = S^5 = I, (RS)^k = (SR)^k \rangle. \tag{22}$$

Then $S_k(\lambda_5)$ is a finite group of order μ_k and the central element T^k has order e_k, where $T = RS$ and

k	1	2	3
e_k	10	5	20
μ_k	10	50	1200

(23)

Lemma 6 *For $k = 1, 2, 3$ let $S_{k,l}(\lambda_5)$ be the group given by*

$$S_{k,l}(\lambda_5) = \langle R, S | R^2 = S^6 = T^{kl} = I, RT^k = T^k R \rangle. \tag{24}$$

Then $S_k(\lambda_5)$ is of order $l\mu_k/e_k$.

Theorem 3 *Let*

$$T_{k,l}(\lambda_5) = \Delta\left(T^{kl}, RT^kRT^{-k}\right). \tag{25}$$

Then $T_{k,l}(\lambda_5)$ is a normal subgroup of $H(\lambda_5)$ with index

$$|H(\lambda_5) : T_{k,l}(\lambda_5)|\, \mu = l\mu_k/e_k < \infty \tag{26}$$

and level

$$n = kl \tag{27}$$

By choosing all possible L such that $l|e_k$, we can find, in total, 12 subgroups $T_{k,l}(\lambda_5)$. The table gives some information about them:

k	1	1	1	2	2	3	3	3	3	3	3	3
l	1	2	5	10	1	5	1	2	4	5	10	20
μ	1	2	5	10	10	50	60	120	240	300	600	1200

(28)

3. Generalization to $q \geq 7$ When $q \geq 7$, λ_q is not "nice" as it is when $q < 7$. It is an algebraic number satisfying th eminimal equation of degree $\varphi(2q)/2$, where φ denotes the Euler function.

As we shall see the two cases when q is odd and even show some differences when we try to find $S_k(\lambda_q)$. Therefore we will have to deal with these two cases separately.

Lemma 7 *Let $g > 6$ be even. For $k = 1, 2$ let*

$$S_k(\lambda_q) = \langle R, S | R^2 = S^q = I, (RS)^k = (SR)^k \rangle. \tag{29}$$

Then $S_k(\lambda_q)$ is a finite group of order μ_k and the central element T^k has order e_k, where $T = RS$ and

k	1	2
e_k	$1q$	q
μ_k	$2q$	$2q^2$

(30)

Proof. A sin the proof of Lemma 1, $RT^k = T^kR$ and therefore T^k is a central element in $S_k(\lambda_q)$ of oreder e_k, where

$$e_k = \frac{2 \cdot 2q}{t} = \frac{4q}{2(k+q) - kq} \tag{31}$$

Now since $t > 0$ and $q > 6$, then

$$k < \frac{2q}{q-2} \tag{32}$$

The fact that $\mu_k = ke_k^2$ for $k = 2$ and that $e_l = q$, $\mu_l = 2q$ can be proved in the same wa as in the proof of Lemma 1.

Lemma 8 *Let $g > 6$ be odd. For $k = 1, 2$ let*

$$S_k(\lambda_q) = \langle R, S | R^2 = S^q = I, (RS)^k = (SR)^k \rangle. \tag{33}$$

Then $S_k(\lambda_q)$ is a finite group of order μ_k and the central element T^k has order e_k, where $T = RS$ and

k	1	2
e_k	$1q$	q
μ_k	$2q$	$2q^2$

(34)

Lemma 9 *Let $q > 6$. For $k = 1, 2$ let $S_{k,l}(\lambda_q)$ be the group given by*

$$S_{k,l}(\lambda_q) = \langle R, S | R^2 = S^q = T^{kl} = I, \; RT^k = T^k R \rangle, \tag{35}$$

where $l | e_k$. Then $S_{k,l}(\lambda_q)$ is of order $l\mu_k / e_k$.

Theorem 4 *Let k and l be chosen as above. Let*

$$T_{k,l}(\lambda_q) = \Delta\left(T^{kl}, RT^k RT^{-k}\right). \tag{36}$$

Then $T_{k,l}(\lambda_q)$ is a normal subgroup of $H(\lambda_q)$ with index

$$\left| H\left(\sqrt{3}\right) : T_{k,l}\left(\sqrt{3}\right)\right| = \mu = l\mu_k / e_k < \infty \tag{37}$$

and level

$$n = kl \tag{38}$$

Proof is exactly the same as the proof of Theorem 1.

By comparing the group representations we can obtain the following result:

$$S_l(\lambda_q) = \begin{cases} T_{l,q}(\lambda_q) & \text{if } q \text{ is even} \\ T_{l,2q}(\lambda_q) & \text{if } q \text{ is odd} \end{cases} \tag{39}$$

For $k = 1$ the relations

$$R^2 = S^q = I, \quad RS = SR \tag{40}$$

hold in the the quotient group $H(\lambda q)/S_1(\lambda_q)$. Therefore this quotient is isomorphic to $C_2 \times C_q$, the direct product of the two cyclic groups of order 2 and order q. It follows that

$$S_1(\lambda_q) \cong H'(\lambda q). \tag{41}$$

4. Enumerating the number of the subgroups $T_{k,l}(\lambda_q)$ Let $q \geq 3$ be given. Let $\eta(q)$ denote the number of $T_{k,l}(\lambda_q)$. Therefore we must consider the conditions on k and l. In the proof of Lemma 7 we obtained the upper and lower bounds for k i. e.

$$1 \leq k \leq \frac{2q}{q-2}. \tag{42}$$

It follows then that all possible values for k are $1 \leq k \leq 5$ for $q = 3$ and $1 \leq k \leq 3$ for $q = 4$ and $1 \leq k \leq 2$ for $q \geq 6$.

Since l can be chosen in the way that $l|e_k$, then the number of l's equals to the sum of the number of divisors of e_k's, denoted by $\sigma(e_k)$ for all possible k's. Therefore

$$\eta(q) = \sum_{1 \leq k < \frac{2q}{q-2}} \sigma(e_k). \tag{43}$$

Recall that $\eta(4) = 10$, $\eta(5) = 12$ and $\eta(6) = 8$. Now let $q > 6$ be even. Then $k = 1$ or $k = 2$ and $e_1 = q = e_2$. Therefore

$$\eta(q) = \sum_{k=1}^{2} \sigma(e_k) = 3\sigma(q). \tag{44}$$

REFERENCES

[1] İ. N. Cangül, *Normal Subgroups of Hecke groups*, Ph. D. Thesis, Southhampton University, 1994,

[2] H. S. M. Coxeter, W. O. J. Moser, *Generators and relations for discrete groups*, Springer Verlag, Berlin, 1957,

[3] M. Newman, *Classification of normal subgroups of the modular groups*, Trans. A. M. S. 126 (1967) p. 267 - 277.

UNIVERSITY OF ULUDAĞ
DEPARTMENT OF MATHEMATICS
Bursa
Turkey