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On the Continuity, Discontinuity and Nonmeasurability of Locally Relatively Continuous Functions

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In paper [2] the authors introduced the notion of a relatively continuous function: A function $f : X \longrightarrow Y$ is called *relatively continuous* at $x \in X$ if, for any open set $V \subset Y$, where $f(x) \in V$, the set $f^{-1}(V)$ is open in the subspace $f^{-1}\left(\overline{V}\right)$. If this condition is satisfied for each $x \in X$, then f is said to be relatively continuous. In paper [5], this notion was generalized in the following way: A function $f : X \longrightarrow Y$ is *locally relatively continuous*[1] if there exists an open base $\mathcal{B}$ for the topology on Y such that $f^{-1}(V)$ is open in the subspace $f^{-1}\left(\overline{V}\right)$ for any $V \in \mathcal{B}$. In [5] the authors also investigated the principal properties of l.r.c. transformations, connected with the continuity and sections of functions, with that they often assumed the considered functions to be connected. The purpose of the following paper is to complete and extend the results included in [5]. Especially, we shall show that l.r.c. functions may have "rather disorderly" properties and, what is more, that the situation is typical (in the topological sense)[2] for this kind

[1]In the further part of the paper we shall use the abbreviation l.r.c. instead of the extended name "locally relatively continuous".

[2]i.e. the set of all l.r.c. functions which do not possess those properties is small in the topological sense - see the considerations in chapter XIII of monograph [1].

of functions. Theorem 2 from paper [5] is also analysed with respect to the possibilities of replacing the connectedness of the considered transformations with the Darboux properties of them.

We shall use the standard notions and notations. By R, Q and N we denote respectively the sets: of all real numbers, rational numbers, positive integers.

A subset $\mathbb{L} \subset X$ is called an *arc* if there exists a homeomorphism $h : [0,1] \longrightarrow \mathbb{L}$. The elements $h(0)$ and $h(1)$ will be called the endpoints of $\mathbb{L}$. The arc with endpoints x and y is denoted by $L(x,y)$. If $\mathbb{L}$ is an arc and $a, b \in \mathbb{L}$, then the symbol $L_{\mathbb{L}}(a,b)$ denotes the arc with endpoints at a and b, which is contained in $\mathbb{L}$.

The open ball with centre at x and radius $r > 0$ will be denoted by $K(x,r)$. $S(x_o, r) = \{x : \varrho(x_o, x) = r\}$ where ϱ denotes the metric in the space considered. The symbols $\overline{A}$ and $\mathrm{Int}\,(A)$ stand for the closure and the interior of A, respectively.

Assume that X is an arbitrary topological space. We say that a nonempty closed set K cuts a space X (onto the sets U and V, between nonempty sets A and B) if $X \setminus K = U \cup V$ where U and V are disjoint, open and nonempty sets (and $A \subset U$ and $B \subset V$).

Let f be a function. If $a \neq b$, we shall write $f(a,b), f(a,b], f^{-1}[a,b)$ etc. instead of $f((a,b)), f((a,b]), f^{-1}([a,b))$, omitting the dispensable double brackets.

By C_f we shall denote the set of all continuity points of f.

If $f : X \times Y \longrightarrow Z$, then by f_x (f^y) we shall denote an $x-$ *section* ($y-$ *section*) of f, i.e. $f_x(t) = f(x,t)$ ($f^y(t) = f(t,y)$).

A function $f : X \longrightarrow Y$ is said to be *quasi-continuous* ([4]) at x if, for each neighbourhood W of $f(x)$ and each neighbourhood U of x, the set $\mathrm{Int}\,(U \cap f^{-1}(W))$ is nonempty. The function f is said to be *quasi-continuous* if it is quasi-continuous at each point of its domain.

The notions and symbols we use, connected with porosity, come from papers [10], [11] and [12].

Let X be a metric space. Let $M \subset X$, $x \in X$ and $S > 0$. Then we denote by $\gamma(x,S,M)$ the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $K(z,r) \subset K(x,S) \setminus M$.

If $p(M,x) = 2 \cdot \limsup_{S \to 0+} \frac{\gamma(x,S,M)}{S} > 0$, then we say that M is *porous* at x.

If there exists $s > 0$ such that $p(M, z) \geq s$ for $z \in X$, then we say that M is *uniformly porous.*

The authors' considerations contained in [5] suggest the question: Do the l.r.c. functions which are not continuous at any point of their domain exist? The answer is positive (Proposition 1). What is more, the authors proved in [5] (Theorem 3): Let X be a locally connected space and let Y and Z be topological spaces. Suppose a function $f : X \times Y \longrightarrow Z$ has continuous x-sections and connected y-sections. Then f is continuous if f is l.r.c. Proposition 1 will show that (under pretty natural assumptions on the spaces considered) there exist l.r.c. functions $f : X \times Y \longrightarrow \mathsf{R}$ discontinuous at every point, whose properties are close to the assertion of the above theorem (the connectedness of y-sections is replaced by the Radakovič property[3]).

Proposition 1 *Let (X, ϱ) be a nonsingleton and connected metric space and Y - an arbitrary metrizable topological space. Then there exists an l.r.c. function $f : X \times Y \longrightarrow \mathsf{R}$ whose all x-sections are continuous and y-sections possess the Radakovič property, such that $C_f = \emptyset$.*

Proof. Let $x_o \in X$. Without loss of generality we may assume that $X \setminus \overline{K(x_o, 1)} \neq \emptyset$. Of course, $S(x_o, \frac{1}{n})$ is a nonempty closed set for $n = 1, 2, \ldots$. Let $\xi : \mathsf{N} \xrightarrow{\text{onto}} \mathsf{Q} \setminus \{0\}$ be a bijection.

In the set $\mathsf{R} \setminus (\{\frac{1}{\mathrm{n}} : \mathrm{n} = 1, 2, \ldots\} \cup \{0\})$ we define an equivalence relation $*$ in the following manner:

$$d * t \iff d - t \in \mathsf{Q}.$$

Let $\mathcal{W}$ be the set of all equivalence classes of the relation $*$. Then there exists a bijection $\eta : \mathcal{W} \xrightarrow{\text{onto}} \mathsf{R} \setminus \mathsf{Q}$.

[3] A function g is *connected* if the image of an arbitrary connected set is also a connected set. If we assume that the closure of the image of a connected set is a connected set (which coincides with the T. Radakovič idea from paper [9]), then we say ([3], [6], [8]) that the considered transformation possesses *the Radakovič property*.

Define a function $f : X \times Y \longrightarrow \mathsf{R}$ by the following formula:

$$f((x,y)) = \begin{cases} 0 & \text{if} \quad x = x_o; \\ \xi(n) & \text{if} \quad x \in S(x_o, \frac{1}{n}) \text{ for } n = 1, 2, \ldots; \\ \eta(\Xi_x) & \text{if} \quad \varrho(x_o, x) \in \Xi_x \in \mathcal{W}. \end{cases}$$

We shall prove that f has the required properties.

To this end, let us notice that

(1) f is an l.r.c. function.

Indeed, let $\beta = \{(p,q) : p,q \in \mathsf{Q} \setminus \{0\}\}$ be a base for the space R and (p_o, q_o) some element of it. Let $n_{p_o}, n_{q_o} \in \mathsf{N}$ be such that $\xi(n_{p_o}) = p_o$ and $\xi(n_{q_o}) = q_o$ and, moreover, let $(z_0, y_o) \in f^{-1}(p_o, q_o)$. Then there exists $\delta_o > 0$ such that $(K(z_0, y_o), \delta_o) \times Y) \cap f^{-1}[p_o, q_o] \subset f^{-1}(p_o, q_o)$. This implies condition (1).

Of course,

(2) f_x is a continuous function for $x \in X$.

Now, we shall show that

(3) f^y is a function which possesses the Radakovič property, for $y \in Y$.

Let $y_0 \in Y$ and let C be a connected subset of $X \times \{y_0\}$. Consider two possibilities:

1) $C \subset S((x_o, y_0), r) \subset X \times \{y_0\}$ for some $r > 0$ or $C = \{(x_o, y_0)\}$. Then, of course, $f^y(C)$ is a singleton.

2) $C \setminus S((x_o, y_0), r) \neq \emptyset$ for any $r > 0$ and $C \neq \{(x_o, y_0)\}$. Let $\alpha \in \mathsf{R} \setminus \mathsf{Q}$. Then there exists $W \in \mathcal{W}$ such that $\eta(W) = \alpha$. In virtue of the connectedness of C, we can easily observe that there exists $k \in C \cap \{(x, y_0) : \varrho(x_o, x) \in W\}$. Then $f(k) = \eta(W) = \alpha$. We have proved that $\mathsf{R} \setminus \mathsf{Q} \subset \mathrm{f(C)}$, what means that $\overline{f(C)} = \mathsf{R}$. This implies condition (3).

In virtue of (1),(2),(3), the proof of Proposition 1 will be finished when we show the discontinuity of the function f at any point. Let $(a,b) \in X \times Y$ and $\delta > 0$. Then $\{x \in K(a,\delta) : \varrho(x_o, x) \in T\} \neq \emptyset$ for any $T \in \mathcal{W}$. This implies that $\mathsf{R} \setminus \mathsf{Q} \subset \mathrm{f(K(a,\delta) \times \{b\})}$, what means that $\mathsf{R} \setminus \mathsf{Q} \subset \mathrm{f(V)}$ for any open neighbourhood V of $(a,b) \in X \times Y$. The proof of Proposition 1 is completed.

The above proposition incites one to pose the next question: If we additionally assume that the considered functions are close to continuity (e.g. quasi-continuous), do there exist Lebesgue nonmeasurable l.r.c. functions? If the answer to this question is positive, is this situation incidental or is it typical (in the topological sense)? The answer is included in the following theorem:

Theorem 1 *In the space C^*_{LR} of bounded quasi-continuous l.r.c. functions $f : \mathsf{R} \longrightarrow \mathsf{R}$, with the metric of uniform convergence, measurable functions (in the Lebesgue sense) constitute a uniformly porous set.*

Proof. Let C^{**}_{LR} denote a subset of C^*_{LR} consisting of nonmeasurable functions and let ϱ^* be a metric of uniform convergence. Let $f \in C^*_{LR}$ be an arbitrary function and $\eta > 0$. Let then $[x_o, y_0]$ be an interval such that $x_o < y_0$ and $f[x_o, y_0] \subset (\alpha_o - \frac{\eta}{3}, \alpha_o + \frac{\eta}{3})$ where α_o is some real number.

Denote by $\mathcal{C}$ the Cantor set with positive Lebesgue measure included in the interval $[x_o, y_0]$ such that $x_o, y_0 \in \mathcal{C}$. The set $\mathcal{C}$ is constructed by "removing" some open intervals from the interval $[x_o, y_0]$. Let A_1 be the interval removed in the first step of the construction of the set $\mathcal{C}$; A_2 - the union of two intervals removed in the second step of the construction of the set $\mathcal{C}$, etc. In this way we shall form a sequence $\{A_n\}_{n=1}^{\infty}$ of open sets, such that $\bigcup_{n=1}^{\infty} A_n = [x_o, y_0] \setminus \mathcal{C}$. Denote by $\mathcal{C}^*$ some nonmeasurable subset of $\mathcal{C}$ such that $x_o, y_0 \notin \mathcal{C}^*$ and $\mathcal{C}^* \cap \bigcup_{n=1}^{\infty} \overline{A_n} = \emptyset$.

Now, let $\mathcal{B}$ denote an open base for the space R, such that $f^{-1}(U)$ is an open set in the subspace $f^{-1}(\overline{U})$ for each $U \in \mathcal{B}$.

Now, we shall define local bases $\mathcal{B}(x)$ at $x \in \mathsf{R}$. Consider the following cases:

A) $x \in (-\infty, \alpha_o - \frac{\eta}{3})$. Then, let

$$\mathcal{B}(x) = \left\{U \in \mathcal{B} : \ x \in U \wedge \overline{U} \cap \left[\alpha_o - \frac{\eta}{3}, +\infty\right) = \emptyset\right\}.$$

B) $x \in \left(\alpha_o + \frac{\eta}{3}, +\infty\right)$. Then, let

$$\mathcal{B}(x) = \left\{U \in \mathcal{B} : \ x \in U \wedge \overline{U} \cap \left(-\infty, \alpha_o + \frac{\eta}{3}\right] = \emptyset\right\}.$$

C) $x = \alpha_o - \frac{\eta}{3}$. Let $U_1 \in \mathcal{B}$ be a set such that $x \in U_1$,

$$U_1 \cap \left(-\infty, \alpha_o - \frac{\eta}{3}\right] \subset \left(\alpha_o - \frac{\eta}{3} - 1, \alpha_o - \frac{\eta}{3}\right],$$

$$U_1\cap\left[\alpha_o-\frac{\eta}{3},+\infty\right)\subset\left[\alpha_o-\frac{\eta}{3},\alpha_o\right)$$

and $f(x_o),f(y_0)\notin U_1$. Put $\beta_1\in U_1\cap\left(\alpha_o-\frac{\eta}{3},+\infty\right)$. Now, let $U_2\in\mathcal{B}$ be a set such that

$$x\in U_2\subset U_1,\ U_2\cap\left(-\infty,\alpha_o-\frac{\eta}{3}\right]\subset\left(\alpha_o-\frac{\eta}{3}-\frac{1}{2},\alpha_o-\frac{\eta}{3}\right],$$

$$U_2\cap\left[\alpha_o-\frac{\eta}{3},+\infty\right)\subset\left[\alpha_o-\frac{\eta}{3},\frac{\beta_1+\alpha_o-\frac{\eta}{3}}{2}\right).$$

Let then $\beta_2\in U_2\cap(\alpha_o-\frac{\eta}{3},+\infty)$. Continuing this process, we shall define a sequence $\{U_n\}_{n=1}^{\infty}\subset\mathcal{B}$ such that $U_{n+1}\subset U_n$ $(n=1,2,\dots)$, $\bigcap_{n=1}^{\infty}U_n=\{\alpha_o-\frac{\eta}{3}\}$, $f(x_o),f(y_0)\notin U_n$ $(n=1,2,\dots)$ and choose a sequence $\{\beta_n\}_{n=1}^{\infty}$ such that $\beta_n\in U_n\cap(\alpha_o-\frac{\eta}{3},\alpha_o)$ and $\beta_n\searrow\alpha_o-\frac{\eta}{3}$. Let us adopt $\mathcal{B}(\alpha_o-\frac{\eta}{3})=\{U_n:\ n=1,2,\dots\}$.
D) $x=\alpha_o+\frac{\eta}{3}$. In a similar way as above we shall define a sequence $\{V_n\}_{n=1}^{\infty}\subset\mathcal{B}$ such that $x\in V_n$ $(n=1,2,\dots)$, $V_{n+1}\subset V_n$, $\bigcap_{n=1}^{\infty}V_n=\{\alpha_o+\frac{\eta}{3}\}$, $V_n\subset(\alpha_o,+\infty)$, $f(x_o),f(y_0)\notin V_n$ $(n=1,2,\dots)$ and choose a sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $\gamma_n\in V_n\cap(\alpha_o,\alpha_o+\frac{\eta}{3})$ and $\gamma_n\nearrow\alpha_o+\frac{\eta}{3}$. Let us adopt $\mathcal{B}(\alpha_o+\frac{\eta}{3})=\{V_n:\ n=1,2,\dots\}$.
E) $x\in(\alpha_o-\frac{\eta}{3},\alpha_o+\frac{\eta}{3})$. Then, let $\mathcal{B}(x)$ be the family of those sets U, for which:

$$x\in U\in\mathcal{B},\ U\subset\left(\alpha_o-\frac{\eta}{3},\alpha_0+\frac{\eta}{3}\right)$$

and the sets $\overline{U}\setminus\{x\}$ and

$$\left(\{\beta_n:\ n\in\mathbb{N}\}\cup\{\gamma_n:\ n\in\mathbb{N}\}\cup\left\{\alpha_o-\frac{\eta}{3},\alpha_o+\frac{\eta}{3},f(x_o),f(y_0)\right\}\right)$$

are disjoint.

The family $\mathcal{B}^*=\bigcup_{x\in\mathbb{R}}\mathcal{B}(x)$ is a base of our topology in R.

Now, define a function $g:\mathbb{R}\longrightarrow\mathbb{R}$ in the following way:

$$g(x)=\begin{cases} f(x) & \text{if } x\in(-\infty,x_o]\cup[y_0,+\infty);\\ \alpha_o+\frac{\eta}{3} & \text{if } x\in C^*;\\ \alpha_o-\frac{\eta}{3} & \text{if } c\in C\setminus(C^*\cup\bigcup_{n=1}^{\infty}\overline{A_n}\cup\{x_o,y_0\});\\ f(x_o) & \text{if } x\in\overline{A_{4n}} \text{ for } n=1,2,...;\\ f(y_0) & \text{if } x\in\overline{A_{4n-1}} \text{ for } n=1,2,...;\\ \beta_{4n-2} & \text{if } x\in\overline{A_{4n-2}} \text{ for } n=1,2,...;\\ \gamma_{4n-3} & \text{if } x\in\overline{A_{4n-3}} \text{ for } n=1,2,... . \end{cases}$$

It is obvious that g is a nonmeasurable and quasi-continuous function.

Now, we shall show that g is an l.r.c. function.

Let $U\in\mathcal{B}^*$. Denote $V=g^{-1}(U)$ and $V^-=g^{-1}(\overline{U})$. Let $v\in V$. Let us analyse the following possibilities:

1) $v\in(-\infty,x_o)\cup(y_0,+\infty)$. Then there exists $\delta_v>0$ such that

$$(v-\delta_v,v+\delta_v)\cap f^{-1}(\overline{U})\subset f^{-1}(U)$$

and $(v-\delta_v,v+\delta_v)\subset(-\infty,x_o)\cup(y_0,+\infty)$. Then

$$(v-\delta_v,v+\delta_v)\cap V^-\subset f^{-1}(U)\cap((-\infty,x_o)\cup(y_0,+\infty))\subset V.$$

2) $v=x_o$. Then $U\in\mathcal{B}$, $f(x_o)=g(x_o)\in U$,

$$(\overline{U}\setminus\{f(x_o)\})\cap\left(\{\beta_n:n\in\mathbb{N}\}\cup\{\gamma_n:n\in\mathbb{N}\}\cup\left\{\alpha_o-\frac{\eta}{3},\alpha_o+\frac{\eta}{3},f(y_0)\right\}\right)=\emptyset$$

and $U\subset(\alpha_o-\frac{\eta}{3},\alpha_o+\frac{\eta}{3})$. Besides, it is known that there exists $\delta_o>0$ such that $(v-\delta_o,v]\cap V^-\subset(v-\delta_o,v]\cap f^{-1}(U)\subset V$ and $v+\delta_o<y_0$. At the same time, $V^-\cap(v,v+\delta_o)\subset g^{-1}(f(x_o))\subset V$, therefore, indeed, $(v-\delta_o,v+\delta_o)\cap V^-\subset V$.

3) $v=y_0$. Our considerations are similar to those in 2).

4) $v\in\overline{A_n}$ $(n=1,2,...)$. Then $V^-\cap(x_o,y_0)=V\cap(x_o,y_0)\subset V$ is an open set in V^-.

5) $v\in C^*$. Then $g(v)=\alpha_o+\frac{\eta}{3}$ and $(x_o,y_0)\cap V^-=(x_o,y_0)\cap V$.

6) $v \in \mathcal{C} \setminus (\mathcal{C}^* \cup \bigcup_{n=1}^{\infty} \overline{A_n} \cup \{x_o, y_0\})$. The reasoning is analogous to that in **5)**.

The arbitrariness of choice of v proves that V is open in V^-.

The proof that g is an l.r.c. function is completed.

Now, we consider the ball $K(g, \frac{\eta}{4})$. Since $\varrho^*(f,g) < \frac{2}{3}\eta$, therefore $K(g, \frac{\eta}{4}) \subset K(f, \eta)$. It is not difficult to notice that if $h \in K(g, \frac{\eta}{4})$, then it is a nonmeasurable function, what proves that $K(g, \frac{\eta}{4}) \subset C^{**}_{LR}$. In turn, this implies that $p(C^*_{LR} \setminus C^{**}_{LR}, f) \geq \frac{1}{2}$, what, by the arbitrariness of the choice of f, ends the proof of the theorem.

In paper [5] the authors proved (Theorem 2) that (under some assumptions concerning the domain of the considered transformations) if f is an l.r.c. and connected function, then it is continuous. It turns out, however, that, in the case of functions defined in R^n, we can replace the connectedness by the Darboux property[4], and even make use of the local Darboux property[5]:

Definition 1 *Let $f : X \longrightarrow Y$ where X, Y are topological spaces. We say that a point $x_o \in X$ is a Darboux point of the third kind of f if, for each arc $L = L(x_o, a)$, the following condition is fulfilled:*
if K is a set such that, for some net $\{x_\sigma\}_{\sigma \in \Sigma} \subset L$ for which $x_o \in \lim_{\sigma\in\Sigma} x_\sigma$, K cuts Y between $\{f(x_o)\}$ and the set $\{f(x_\sigma) : \sigma \in \Sigma\} \cup acp_{\sigma\in\Sigma} f(x_\sigma)$,[6]
then $K \cap f(L_L(x_o, x_\sigma)) \neq \emptyset$ for any $\sigma \in \Sigma$.

Theorem 2 *Let $f : \mathsf{R}^n \longrightarrow Y$ be an l.r.c. function where Y is a connected regular topological space. Then the following conditions are equivalent:*
(i) x_o is a Darboux point of the first kind of f.

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[4] i.e. (see [7]), the image of every arc is a connected set.

[5] In the theorem below we use the definition of a Darboux point of the third kind only, therefore we shall quote it in full here. The definitions of Darboux points of the first and the second kinds are pretty long([7], [8]), so we shall not quote them - in the proof of theorem, for this kind of points, we make direct use of the result included in the papers cited above.

[6] By $acp_{\sigma\in\Sigma} f(x_\sigma)$ we denote the set of all accumulation points of $\{f(x_\sigma)\}_{\sigma\in\Sigma}$.

(ii) x_o *is a Darboux point of the second kind of* f.
(iii) x_o *is a Darboux point of the third kind of* f.
(iv) $x_o \in C_f$.

Proof. In view of the results included in papers [7] and [8], it suffices to prove the implication *(iii)*⇒*(iv)*.

Suppose that $x_o \notin C_f$. Then there exists an open neighbourhood V of a point $f(x_o)$, such that

$$f(K(x_o,\delta)) \setminus V \neq \emptyset \text{ for any } \delta > 0.$$

This means that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$\varrho(x_o, x_n) \searrow 0 \quad \text{and} \quad f(x_n) \notin V, \quad \text{for} \quad n \in \mathsf{N}$$

Let, for an arbitrary $n = 1, 2, ...,$ $t_n \in (\varrho(x_o, x_n), \varrho(x_o, x_{n+1}))$. Denote I_n^* (I_n^{**}) an interval, one endpoint of which is x_n (x_{n+1}), but the other endpoint lies on the sphere $S(x_o, t_n)$, with that we demand that the lenght of I^* (I^{**}) is equal to $\varrho(S(x_o, t_n), x_n)$ ($\varrho(S(x_o, t_n), x_{n+1})$). Moreover, let $I_n^{***} \subset S(x_o, t_n)$ be an arc which endpoints belong to I_n^* and I_n^{**}.

Let $\mathrm{L} = \{x_o\} \cup \bigcup_{n=1}^{\infty} (I_n^* \cup I_n^{**} \cup I_n^{***})$. Then L is an arc, one endpoint of which is x_o.

Let $\mathcal{B}$ be a base of the space Y such that
(4) $f^{-1}(U)$ is an open set in $f^{-1}(\overline{U})$, for $U \in \mathcal{B}$. Choose $U_o \in \mathcal{B}$ such that $f(x_o) \in U_o \subset \overline{U_o} \subset V$ (this choice is possible because X is a T_3-space).

Denote $F = Fr(U_o)$. Then $F \neq \emptyset$ in virtue of the connectedness of Y, and this means that F cuts Y between $\{f(x_o)\}$ and the set $\{f(x_n) : n \in \mathsf{N}\} \cup \mathrm{acp}(\mathrm{f}(\mathrm{x_n}))$. Since x_o is a Darboux point of the third kind of f, we can infer that $x_o \in \overline{f^{-1}(F)}$. This fact implies (according to the fact that U_o is an open set) that $f^{-1}(U_o)$ is not an open set in $f^{-1}(\overline{U_o})$, what contradicts to (1). The obtained contradiction ends the proof of the theorem.

REFERENCES

[1] Bruckner A.M., *Differentiation of real functions*, Springer-Verlag Berlin-Heidelberg-New York (1978).

[2] Chew J., Tong J., *Some remarks on weak continuity*, Amer. Math. Month. 98 (1991), 931-934.

[3] Farkova J., *Über Darbouxsche Funktionen*, Math. Čas. 20 (1970), 185-194.

[4] Kempisty S., *Sur les fonctions quasicontinues*, Fund. Math. 19 (1932), 184-197.

[5] Mimna R.A., Rose D.A., *On local relative continuity*, Real Anal. Exch. 20.2 (1994-95), 823-830.

[6] Mišik L., *Über die Eigenschaft von Darboux und einiger Klassen von Funktionen*, Rev. Roum. Math. Pures et Appl. 11 (1966), 411-430.

[7] Pawlak R.J., *Przekształcenia Darboux*, habilitation thesis (1985), 1-148.

[8] Pawlak R.J., *On local characterization of Darboux functions*, Comm. Math. 27 (1988), 283-299.

[9] Radakovič T., *Über Darbouxsche und stetige Funktionen*, Mont. Math. Phys. 38 (1931), 111-122.

[10] Renfro D., *On various porosity notions in the literature*, manuscript from private letter.

[11] Zajíček L., *Sets of σ-porosity and sets of σ-porosity(q)*, Časopis Pěst. Mat. 101 (1976), pp.350-359.

[12] Zajíček L., *Porosity and σ-porosity*, Real Anal. Ex. 13 (1987-88), pp.314-350.

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