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The Sum of 1-improvable Functions

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Definition 1 *For each function $f : \mathbb{R} \longrightarrow \mathbb{R}$, by $L(f)$ we denote the set of all points at which there exists a limit of the function f. Furthermore, let*

$$C(f) = \left\{ x \in L(f);\ \lim_{t \to x} f(t) = f(x) \right\};$$
$$U(f) = \left\{ x \in \mathbb{R};\ \lim_{t \to x} f(t) \neq f(x) \right\}.$$

We define the functions $f_{(\alpha)}$ for all ordinal numbers.

Definition 2 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and let $f_{(0)}(x) = f(x)$ for each $x \in \mathbb{R}$. For every ordinal number α, let*

$$f_{(\alpha)}(x) = \begin{cases} f(x) & \text{if} \quad \left\{\gamma < \alpha;\ x \in U\left(f_{(\gamma)}\right)\right\} = \emptyset, \\ \lim_{t\to x} f_{(\gamma_0)}(t) & \text{if} \quad x \in U\left(f_{(\gamma_0)}\right), \\ & \text{where } \gamma_0 = \min\left\{\gamma < \alpha;\ x \in U\left(f_{(\gamma)}\right)\right\}. \end{cases}$$

Definition 3 *For each ordinal number α, we denote*

$$\mathcal{A}_\alpha = \left\{ f : \mathbb{R} \longrightarrow \mathbb{R};\ C\left(f_{(\alpha)}\right) = \mathbb{R} \right\}.$$

We can state the following remark:

Remark 1 *The family* $(\mathcal{A}_\alpha)_{\alpha\geq 0}$ *has the following properties:*

(1) $\mathcal{A}_0$ *is the family of all continuous functions on* D*;*

(2) *for each ordinal number* $\alpha < \omega_1$, $\bigcup_{0\leq\beta<\alpha} \mathcal{A}_\beta \subset \mathcal{A}_\alpha$.

Definition 4 *If a function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *belongs to* $\mathcal{A}_\alpha \setminus \left(\bigcup_{0\leq\beta<\alpha} \mathcal{A}_\beta\right)$, *then it will be called* **an α-improvable discontinuous function**.

It is easy to see the following remark:

Remark 2 *Let* $W \subset \mathbb{R}$ *and let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be the characteristic function of the set* W *such that* $f \in \mathcal{A}_\alpha \setminus \bigcup_{\beta<\alpha} \mathcal{A}_\beta$ *for some ordinal number* $\alpha < \omega_1$. *Then each function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *such that for each* $x \in \mathbb{R}$, $0 \leq g(x) < f(x)$ *belongs to* $\mathcal{A}_\alpha$.

For any subset K of $\mathbb{R}$ by K^d we shall denote the set of all accumulation points of the set K.

Theorem 1 *For each ordinal number* $\alpha < \omega_1$ *there exist functions* f, g *belonging to* $\mathcal{A}_1$ *such that* $f + g \in \mathcal{A}_\alpha$.

Proof. For each set $A \subset \mathbb{R}$ and $a, b \in \mathbb{R}$ we denote $aA + b = \{ax + b; x \in A\}$.

Let $\alpha < \omega_1$ be an ordinal number. By the transfinite induction, we shall define a sequence of sets $(W_\beta)_{\beta\leq\alpha}$ in the following way: let $W_0 = \{0\}$ and let $W_1 = \left\{\frac{1}{2^n};\ n \in \mathbb{N}\right\}$ and, for each ordinal number β (where $3 \leq \beta \leq \alpha$).

1. if $\beta = \gamma + 2$, where β is an ordinal number, then put

$$W_\beta = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} W_\gamma + \frac{1}{2^n}\right),$$

2. if β is a limit ordinal number, then we can choose a sequence $(\beta_n)_{n=1}^{\infty}$ of ordinal numbers such that $\lim_{n\to\infty}\beta_n = \beta$ and, for each $n \in \mathbb{N}$, $\beta_n < \alpha$, thus we put

$$W_\beta = \bigcup_{n=1}^{\infty}\left(\frac{1}{2^n}W_{\beta_n} + \frac{1}{2^n}\right),$$

3. if $\beta = \gamma + 1$, where γ is a limit ordinal number, then

$$W_\beta = W_\gamma \cup \{0\}.$$

Notice that $W_\gamma \cap W_\xi = \emptyset$ whenever $\gamma \neq \xi$ and $W_\gamma = W_{\gamma+1}^d$ for each $\gamma \leq \alpha$.

Now, we define a sequence of functions $(f_\beta)_{\beta<\alpha}$ and $(g_\beta)_{\beta<\alpha}$ in the following way: let $f_0(x) = 0$ and $g_0(x) = 0$ for each $x \in \mathbb{R}$; let f_1 be the characteristic function of the set W_0 and $g_1 = g_0$ and let $f_2 = f_1$ and $g_2 = \frac{1}{2}\chi_{W_1}$. Assume that β is an ordinal number such that $3 \leq \beta < \alpha$ and assume that, for each ordinal number γ (where $3 \leq \gamma < \beta$), we have defined functions f_γ, g_γ, then

1. if $\beta = \gamma + 2$, where γ is an ordinal number, we have to consider two possibilities:
   - let β be an odd number, then put

$$f_\beta(x) = \begin{cases} \frac{1}{2^n}g_{\gamma+1}(2^n x - 1) & \text{if } x \in \frac{1}{2^n}W_\gamma + \frac{1}{2^n}, \\ f_{\gamma+1}(x) & \text{if } x \in \mathbb{R} \setminus W_{\gamma+1} \end{cases}$$

   and $g_\beta = g_{\gamma+1}$;
   - let β be an even ordinal number, then put $f_\beta = f_{\gamma+1}$ and

$$g_\beta(x) = \begin{cases} \frac{1}{2^n}f_{\gamma+1}(2^n x - 1) & \text{if } x \in \frac{1}{2^n}W_\gamma + \frac{1}{2^n}, \\ g_{\gamma+1}(x) & \text{if } x \in \mathbb{R} \setminus W_{\gamma+1}; \end{cases}$$

2. if β is a limit ordinal number, then we put

$$f_\beta(x) = \begin{cases} \frac{1}{2^n}f_{\beta_n}(x) & \text{if } x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}), \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_\beta(x) = \begin{cases} \frac{1}{2^n} g_{\beta_n}(x) & \text{if } x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}), \\ 0 & \text{otherwise,} \end{cases}$$

where the sequence $(\beta_n)_{n=1}^\infty$ was chosen when we were constructing the set W_β;

3. if $\beta = \gamma + 1$, where γ is a limit ordinal number, then put

$$f_\beta(x) = \begin{cases} 1 & \text{if } x \in W_0, \\ f_\gamma & \text{otherwise} \end{cases}$$

and $g_\beta = g_\gamma$.

First we show that $f_\alpha \in A_1$.

Observe that

$$\{x \in \mathbb{R}\,; f(x) > 0\} \subset \bigcup_{\beta<\alpha} W_\beta.$$

Since $\mathrm{cl}\left(\mathbb{R} \setminus \bigcup_{\beta<\alpha} W_\beta\right) = \mathbb{R}$ and $\mathbb{R} \setminus \bigcup_{\beta<\alpha} W_\beta \subset \{x \in \mathbb{R};\ f(x) = 0\}$, we have

$$\left\{x \in \mathbb{R};\ \liminf_{t\to x} f_\alpha(t) = 0\right\} = \mathbb{R}.$$

Consider three possibilities:

1. Let $\alpha = \beta + 2$, where β is an ordinal number. Let $x \in \mathbb{R}$. If there exists $\gamma < \alpha$ such that γ is an odd ordinal number and $x \in W_\gamma$, then $f_\alpha(x) > 0$. Thus there exists a sequence $(x_n)_{n=1}^\infty \subset W_{\gamma+1}$ such that $\lim_{n\to\infty} x_n = x$. Since for each $n \in \mathbb{N}$, $f_\alpha(x_n) = 0$, we have $\lim_{n\to\infty} f_\alpha f(x_n) = 0$. By the definition of the function f_α, we infer that for each sequence $(x_n)_{n=1}^\infty$ in $\bigcup_{\gamma+1<\xi<\alpha} W_\xi$, $\lim_{n\to\infty} f_\alpha(x_n) = 0$. Thus there exists $\lim_{t\to x} f_\alpha(t)$ and $\lim_{t\to x} f_\alpha(t) = 0$. Hence $x \in U(f_\alpha)$ and $x \in C\left((f_\alpha)_{(1)}\right)$.

   If there exists $\gamma < \alpha$ such that γ is an even ordinal number and $x \in W_\gamma$, then $f_\alpha(x) = 0$. Then since $W_\gamma = W_{\gamma+1}^d$, there exists a sequence $(x_n)_{n=1}^\infty \subset W_{\gamma+1}$ such that $\lim_{n\to\infty} x_n = x$. Since for each $n \in \mathbb{N}$, $f(x_n) > \frac{1}{2^n} > 0$ for some $n \in \mathbb{N}$, we have

$\lim_{n\to\infty} f_\alpha(x_n) > 0$. Since $\gamma + 1$ is an odd ordinal number, for each $n \in \mathbb{N}$, $x_n \in U(f_\alpha)$, hence $x \in C\left(\left(f_\alpha\right)_{(1)}\right)$.

If $x \in \mathbb{R} \setminus \bigcup_{\gamma<\alpha} W_\gamma$, then there exists no ordinal number $\gamma < \alpha$ such that $x \in W_\gamma^d$. Thus $\lim_{t\to x} f_\alpha(t) = 0$. Hence $x \in C(f)$.

Thus $f_\alpha \in \mathcal{A}_1$.

2. Let α be a limit ordinal number. Then we can show analogously that for each $n \in \mathbb{N}$, the function $(f_\alpha)_{|[\frac{1}{2^{n+1}},\frac{1}{2^n})} \in \mathcal{A}_1$, hence $f_\alpha \in \mathcal{A}_1$.

3. Let $\alpha = \beta + 1$, where β is a limit ordinal number. Then we can show similarly to above that $(f_\alpha)_{|(\mathbb{R}\setminus\{0\})} \in \mathcal{A}_1$. Since for each sequence $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = 0$ and for each $n \in \mathbb{N}$, $f(x_n) > 0$ we have $\lim_{n\to\infty} f_\alpha(x_n) = 0$, so $0 \in U(f_\alpha)$. Thus $f_\alpha \in \mathcal{A}_1$.

Similarly to above we can show that $g_\alpha \in \mathcal{A}_1$.

Put $h_\alpha = f_\alpha + g_\alpha$.

Analogously to the proof of Theorem 13 (see [1]) we can show that $\chi_{W_\alpha} \in \mathcal{A}_\alpha$.

Since for each $x \in \mathbb{R}$, $0 \le h_\alpha(x) \le 2\chi_{W_\alpha}(x)$, we have by Remark 1 that $h_\alpha \in \mathcal{A}_\alpha$. It is easy to see that $h_\alpha \notin \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. Thus the proof is completed.

REFERENCES

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