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On the Function of Q-oscillation

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Throughout this paper we shall use the following denotations, facts and definitions.

$\mathbb{R}$ will denote the set of all real numbers.

Definition 1 *Let $\mathcal{B}_0^+ \subset 2^{\mathbf{R}}$ be a nonempty family of nonempty sets fulfilling the following conditions:*

(1) *if $B \in \mathcal{B}_0^+$, then for every $t > 0$, $B \cap (0,t) \in \mathcal{B}_0^+$,*

(2) *$B_1 \cup B_2 \in \mathcal{B}_0^+$ if and only if $B_1 \in \mathcal{B}_0^+$ or $B_2 \in \mathcal{B}_0^+$.*

For every set $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ we shall write

$$A + x = \{y \in \mathbb{R} : \exists_{a\in A}(y = a + x)\}, \quad -A = \{y \in \mathbb{R} : (-y \in A)\}.$$

Then the family $\mathcal{B}_0^-$ is defined as

$$\mathcal{B}_0^- = \left\{B \subset \mathbb{R} : -B \in \mathcal{B}_0^+\right\}.$$

For each $x \in \mathbb{R}$ let

$$\mathcal{B}_x^+ = \left\{B \subset \mathbb{R} : (B - x) \in \mathcal{B}_0^+\right\}, \mathcal{B}_x^- = \left\{B \subset \mathbb{R} : (-B + 2x) \in \mathcal{B}_x^+\right\}$$

and $\mathcal{B}_x = \mathcal{B}_x^+ \cup \mathcal{B}_x^-$. Now let $\mathcal{B} = \{\mathcal{B}_x\}_{x\in\mathbf{R}}$.

Definition 2 *A number g is called a $\mathcal{B}$-limit number of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at a point x_0 if for every positive number ε the set*

$$\{x \in \mathbb{R} : |f(x) - g| < \varepsilon\} \in \mathcal{B}_{x_0}.$$

By $L_{\mathcal{B}}(f, x)$ we shall denote the set of all $\mathcal{B}$-limit numbers of the function f at the point x.

For every function f and every point $x \in \mathbb{R}$ there exists at least one $\mathcal{B}$-limit number of f at x. For every $f : \mathbb{R} \longrightarrow \mathbb{R}$ and every $x \in \mathbb{R}$ the set $L_{\mathcal{B}}(f, x)$ is closed.

Definition 3 *We say that a family $\mathcal{B}$ fulfils the condition M_0, if for every $x_0 \in \mathbb{R}$ and a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \searrow x_0$ and for every sequence $(B_n)_{n=1}^{\infty}$ such that $B_n \in \mathcal{B}_{x_n}$ the set $\bigcup_{n=1}^{\infty} B_n$ belongs to the family $\mathcal{B}_{x_0}^{+}$.*

Definition 4 *We say that a family $\mathcal{B}$ fulfils the condition M_1, if for every $x_0 \in \mathbb{R}$ and a set $E \in \mathcal{B}_{x_0}^{+}$ and for every family of sets $\{B_x\}_{x \in E}$ such that $B_x \in \mathcal{B}_x$ for $x \in E$ the set $\bigcup_{x \in E} B_x$ belongs to the family $\mathcal{B}_{x_0}^{+}$.*

One can see that each family $\mathcal{B}$ fulfilling condition M_0 fulfils also condition M_1.

For a bounded function $f : \mathbb{R} \longrightarrow \mathbb{R}$ let us write:

$$m_{\mathcal{B}}(f, x) = \min L_{\mathcal{B}}^{\star}(f, x),$$

$$M_{\mathcal{B}}(f, x) = \max L_{\mathcal{B}}^{\star}(f, x),$$

where $L_{\mathcal{B}}^{\star}(f, x) = L_{\mathcal{B}}(f, x) \cup \{f(x)\}$. We shall say that a function f is upper $\mathcal{B}$-semicontinuous (lower $\mathcal{B}$-semicontinuous) at a point x_0 if

$$M_{\mathcal{B}}(f, x) \leq f(x) \quad (m_{\mathcal{B}}(f, x) \geq f(x)).$$

From theorem 14 in article [2] we infer the following characterization: for an arbitrary bounded function f the function $M_{\mathcal{B}}(f)$ is upper $\mathcal{B}$-semicontinuous if and only if the family $\mathcal{B}$ fulfils condition M_1; and similarly, for an arbitrary bounded function f the function $m_{\mathcal{B}}(f)$ is lower $\mathcal{B}$-semicontinuous if and only if the family $\mathcal{B}$ fulfils condition M_1.

The qualitative limit numbers are obtained from the family $\mathcal{Q}$ defined as follows: a set E belongs to the family $\mathcal{Q}_0^+$ if for every $t > 0$ the set $E \cap (0, t)$ is of the second category. The family $\mathcal{Q}$ fulfils condition M_0. $\mathcal{Q}$-oscillation $\mathcal{Q} - \omega_f(x)$ of a bounded function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at a point $x \in \mathbb{R}$ is defined as follows:

$$\mathcal{Q} - \omega_f(x) = M_{\mathcal{Q}}(f, x) - m_{\mathcal{Q}}(f, x).$$

The symbol $\mathcal{T}$ will denote here the natural topology on the set of all real numbers, $\mathcal{C}$ – the class of all subsets of $\mathbb{R}$ which are of the first category. Let $\mathcal{T}_{\mathcal{Q}}$ denote the following topology on $\mathbb{R}$.

$$\mathcal{T}_{\mathcal{Q}} = \{U \setminus H \,:\, U \in \mathcal{T},\ \ H \in \mathcal{C}\}.$$

This topology is sometimes called the qualitative topology. Now let us observe that for an arbitrary bounded function f, f is upper $\mathcal{Q}$-semicontinuous if and only if for each $a \in \mathbb{R}$

$$E_a = \{x \in \mathbb{R} : f(x) > a\} \in \mathcal{T}_{\mathcal{Q}}.$$

Now one can notice the following properties for each bounded function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

[1] the set $\Omega_f(y) = \{x \in \mathbb{R} : \mathcal{Q} - \omega_f(x) \geq y\}$ is $\mathcal{T}_{\mathcal{Q}}$-closed for each $y \in \mathbb{R}$;

[2] if $y_1 < y_2$ then $\Omega_f(y_2) \subset \Omega_f(y_1)$;

[3] the set $\bigcup_{y \in \mathbb{R}} (\Omega_f(y) \times \{y\})$ is $\mathcal{T}_{\mathcal{Q}} \times \mathcal{T}$-closed on the plane.

We say that D is a set of the second category at a point x if the set $(x - \delta, x + \delta) \cap D$ is of the second category for each $\delta > 0$. Let D_q denote the set of all points of the second category of the set D. It is known that for each set $D \subset \mathbb{R}$ the set $D \setminus D_q$ is of the first category. It follows from those facts that every set E can be represented in the form of the union of two sets, the first of which is consising of all points of the second category of E, and the second one is of the first category.

Let now $\{\Omega(y)\}_{y\in[0,1]}$ be a nonempty family of subsets of $\mathbb{R}$ such that:

(α_1) the set $\Omega(y)$ is $\mathcal{T}_Q$-closed for each $y \in [0,1]$;

(α_2) if $0 \le y_1 < y_2 \le 1$ then $\Omega(y_2) \subset \Omega(y_1)$;

(α_3) the set $\bigcup_{y\in\mathbb{R}} (\Omega(y) \times \{y\})$ is $\mathcal{T}_Q \times \mathcal{T}$-closed on the plane;

(α_4) $\Omega(0) = \mathbb{R}$.

For each $y \in [0,1]$ let

$$\Omega(y) = A(y) \cup B(y),$$

where $A(y)$ consists of all points of the second category of $\Omega(y)$ and $B(y) = \Omega(y) \setminus A(y)$. We shall prove the following theorem:

Theorem 1 *For each family $\{\Omega(y)\}_{y\in[0,1]}$ fulfilling conditions (α_1) – (α_4) there exists a function $f : \mathbb{R} \longrightarrow [0,1]$ such that*

$$\Omega_f(y) = \Omega(y) \ \text{ for each } \ y \in [0,1].$$

Proof. Notice, that if for some $y' \in (0,1]$, $x \in A(y')$, then $x \in A(y)$ for each $y \in [0, y')$. If $x \in B(y')$, x need not belong to each $B(y)$ for $y \in [0, y')$. However if $x \in B(y'')$ for some $y'' < y'$, then $x \in B(y)$ for each $y \in (y'', y')$. For each $a \in \mathbb{R}$ let us define the set B_a as follows:

$$B_a = \{y \in [0,1] \,:\, a \in B(y)\}.$$

Let F be the set of all those points a from $\mathbb{R}$ for which B_a is a nondegenerate interval.

Lemma 1 *The set F is of the first category.*

Proof of Lemma. First we shall show that for $x \in F$, x is not a point of the second category of the set F. Let x be an arbitrary point of the set f. Choose an arbitrary positive number h. Let

$$y_0 = \inf\{y \in [0,1] : [(x-h, x+h) \setminus A(y)] \cap F \neq \emptyset\}.$$

Consider the following sequence of sets:

$$W_1 = \left[(x-h, x+h) \setminus A\left(y_0 + \frac{1-y_0}{2}\right)\right] \cap F,$$

$$W_2 = \left[\bigcup_{k=1}^{2^2}\left((x-h, x+h) \setminus A\left(y_0 + \frac{k(1-y_0)}{2^2}\right)\right)\right] \cap F,$$

....

$$W_{2n} = \left[\bigcup_{k=1}^{2^{2n}}\left((x-h, x+h) \setminus A\left(y_0 + \frac{k(1-y_0)}{2^{2n}}\right)\right)\right] \cap F,$$

....

It is easy to verify that

$$(x-h, x+h) \cap F = W_1 \cup \bigcup_{n=1}^{\infty} W_{2n}.$$

It follows from the above equality that the set $(x-h, x+h) \cap F$ is of the first category.

Now we can return to the proof of the theorem. For each x from the set F let $h_x > 0$ be chosen in such a way that the above equality holds.

First, observe that

$$F = \bigcup_{x \in F} [(x-h_x, x+h_x) \cap F].$$

From Lindelöf's theorem it follows that

$$F = \bigcup_{x \in F'} [(x-h_x, x+h_x) \cap F],$$

where F' is a certain countable subset of F.

We shall apply the following proposition: For every subset E of the set $\mathbb{R}$ there exist two sets E_1 and E_2 such that

(a) $E_1 \subset E, \quad E_2 \subset E, \quad E_1 \cup E_2 = E, \quad E_1 \cap E_2 = \emptyset,$
(b) $(E_1)_q = E_q, \quad (E_2)_q = E_q.$

Now let

$$A = \{x \in \mathbb{R} : \sup\{y \in [0,1] : x \in A(y)\} > 0\}.$$

We are ready to define a function $f : \mathbb{R} \longrightarrow [0,1]$ as follows:

$$f(x) = \begin{cases} \sup\{y \in [0,1] : x \in \Omega(y)\} & \text{for} \quad x \in A_1 \cup F, \\ 0 & \text{for} \quad x \in A_2 \cup (\mathbb{R} \setminus F), \end{cases}$$

Of course, $\Omega_f(y) \subset \Omega(y)$. We shall show the converse inclusion. Let $y_0 \in (0,1]$ and $x \in \Omega(y_0)$. Consider two possibilities:
(I) $x \notin F$.
(II) $x \in F$.

In the first case, let $y_0' = \sup\{y \in [0,1] : x \in A(y)\}$. Condition (α_3) implies that $x \in \Omega(y_0)$ and $y_0' = \max L_{\mathcal{B}}(f,x) = \max L^\star_{\mathcal{B}}(f,x) = f(x)$. Now $\mathcal{Q} - \omega_f(x) = y_0'$, hence $x \in \Omega_f(y_0')$. Since $y_0 \le y_0'$, then $x \in \Omega_f(y_0)$.

In the second case $y_0' < f(x)$ and from condition (α_3) we infer that $x \in \Omega(f(x))$. Thus $\mathcal{Q} - \omega_f(x) = f(x) > y_0$ and consequently $x \in \Omega_f(y_0)$.

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