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On algebra generated by derivatives of interval functions

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0.1 Introduction.

In 1982 D. Preiss [4] proved the following

Theorem 0.1.1 *Whenever $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of the first class, there are functions f, g and h possessing finite derivative everywhere such that $u = f' \cdot g' + h'$. Moreover, one can find such a representation that g' is bounded and h' is Lebesgue and in case u is bounded, such that f' and h' are also bounded,*

which was the solution of A. Bruckner's problem concerning the algebra generated by derivatives (it is exactly the first class).

In this article we generalize this theorem changing the domain of u. However, we obtain the generalization only for bounded functions. In the proof we use the Preiss's method.

0.2 Preliminaries.

In this section we develop notation and state some known results to which we shall refer.

The real line $(-\infty, +\infty)$ we denote by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. Throughout this article m is fixed positive integer and the word function means mapping from $\mathbb{R}^m$ into $\mathbb{R}$ unless otherwise explicitly stated.

A function is said to be in the first class of Baire ($\mathcal{B}^1$) if it is a pointwise limit of a sequence of continuous functions (with respect to natural topology). We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $\mathbb{R}^m$. The Euclidean distance of two points in $\mathbb{R}^m$, of a point in $\mathbb{R}^m$ and a subset of $\mathbb{R}^m$ and of two subsets of $\mathbb{R}^m$ are denoted respectively by $\varrho(x,y)$, $\varrho(x,A) = \inf\{\varrho(x,y) : y \in A\}$ and $\varrho(A,B) = \inf\{\varrho(x,y) : x \in A, y \in B\}$. For each $A \subset \mathbb{R}^m$ we denote by $|A|$ its outer Lebesgue measure, by $\operatorname{diam} A$ its diameter and by χ_A its characteristic function. A differentiation basis is a pair $(\mathcal{I}, \Rightarrow)$, where $\mathcal{I} \subset \mathcal{L}$ is composed of sets of positive finite Lebesgue measure and $\Rightarrow$ is a relation (called the convergency relation) between sequences of elements of $\mathcal{I}$ and points of $\mathbb{R}^m$, such that the following two conditions hold:

1. for each $x \in \mathbb{R}^m$ there is a sequence $\{I_n : n \in \mathbb{N}\}$ of elements of $\mathcal{I}$ convergent to x.

2. each subsequence of a sequence convergent to some x is also convergent to x.

If $P : \mathcal{I} \longrightarrow \mathbb{R}$ then the notation $\lim_{I \Rightarrow x} P(I) = a$ means "for each sequence $\{I_n : n \in \mathbb{N}\} \subset \mathcal{I}$, if $\{I_n : n \in \mathbb{N}\} \Rightarrow x$, then $\lim_{n\to\infty} P(I_n) = a$." We call function f a derivative (with respect to the differentiation basis $(\mathcal{I}, \Rightarrow)$) iff there exists an additive function $F : \mathcal{I} \longrightarrow \mathbb{R}$ such that $\lim_{I \Rightarrow x} F(I)/|I| = f(x)$ for each $x \in \mathbb{R}^m$. We say that f is a Lebesgue function (w.r.t. $(\mathcal{I}, \Rightarrow)$) iff $\lim_{I \Rightarrow x} \int_I |f(t) - f(x)|\, dt/|I| = 0$ for each $x \in \mathbb{R}^m$. (We recall that each Lebesgue function is approximately continuous and each bounded approximately continuous functions is Lebesgue.) Point $x \in \mathbb{R}^m$ is a density point of $A \in \mathcal{L}$ (w.r.t. $(\mathcal{I}, \Rightarrow)$) iff $\lim_{I \Rightarrow x} |A \cap I|/|I| = 1$. By $A \prec B$ we denote that $A \subset B$ and each $x \in A$ is a density point of B. We call function f approximately continuous iff for each $x \in \mathbb{R}^m$ and each $\varepsilon > 0$, x is a density point of $\{y \in \mathbb{R}^m : |f(y) - f(x)| < \varepsilon\}$. We denote by $a \vee b$ ($a \wedge b$) not smaller (not greater) of real numbers a and b, respectively. If f is any function and $x \in \mathbb{R}^m$ then by $\omega(f, x) = \inf\{\sup\{|f(y) - f(z)| : |y - x| < \varepsilon, |z - x| < \varepsilon\} : \varepsilon > 0\}$ we denote the oscillation of f at x; by $\|f\|$ we denote $\sup\{|f(x)| : x \in \mathbb{R}^m\}$. By sgn we denote the sign function.

Till the end of this article $(\mathcal{I}, \Rightarrow)$ denotes so called ordinary differentiation basis (i.e., $\mathcal{I}$ is the family of all m-dimensional intervals and $\{I_n : n \in \mathbb{N}\} \Rightarrow x$

iff $\{x\} = \bigcap_{n=1}^{\infty} I_n$, $\lim_{n\to\infty} \operatorname{diam} I_n = 0$ and $\limsup_{n\to\infty} (\operatorname{diam} I_n)^m / |I_n| < \infty$) and terms derivative, Lebesgue function, approximately continuous function and density point are used with respect to this basis.

The following three theorems are due to M. Chaika [1] and Z. Grande [2].

Theorem 0.2.1 *Assume that $A \in \mathcal{L}$, F is closed and $F \prec A$. Then there is a closed set $B \subset A$ such that $F \prec B$.*

Theorem 0.2.2 *Assume that E, F are disjoint sets of type G_δ, such that the sets $\mathbb{R}^m \setminus E$ and $\mathbb{R}^m \setminus F$ contain only the density points of themselves. Then there exists an approximately continuous function f such that:*

- *$f(x) = 0$, if $x \in E$,*
- *$f(x) = 1$, if $x \in F$,*
- *$0 < f(x) < 1$, if $x \notin (E \cup F)$.*

Theorem 0.2.3 *Whenever $f \in B^1$ and $E \subset \mathbb{R}^m$ is a null set, there is an approximately continuous function g such that $f(x) = g(x)$ for $x \in E$.*

0.3 Auxiliary lemmas and main results.

Lemma 0.3.1 *Suppose that $B \in \mathcal{L}$, $F_1, \ldots, F_n$ are pairwise disjoint, closed subsets of B, such that $F_i \prec B$ for $i = 1, \ldots, n$ and $c_1, \ldots, c_n \in \mathbb{R}$. Then there is a Lebesgue function f such that $f(x) = c_i$ if $x \in F_i$, $i = 1, \ldots, n$, $f(x) = 0$ if $x \notin B$ and $\|f\| \leq \max\{c_i : i = 1, \ldots, n\}$.*

Proof. For $i = 1, \ldots, n$ we put $d_i = \{\varrho(x, F_j) : j = 1, \ldots, n, j \neq i\}$ and $A_i = \{x \in B : \varrho(x, F_i) < d_i\}$. Then $F_i \prec A_i$, since if $x \in F_i$, $x \in I \in \mathcal{I}$ and $\operatorname{diam} I < d_i$, then $I \cap A_i = I \cap B$. Let $B_i \subset A_i$ be closed and such that $F_i \prec B_i$ (Theorem 0.2.1). Find a G_δ set C_i containing all points of B_i that are not points of density of B_i, such that $|C_i| = 0$ and $C_i \cap F_i = \emptyset$. Apply Theorem 0.2.2 to find an approximately continuous function f_i such that:

- $f_i(x) = 0$, if $x \in \mathbb{R}^m \setminus (B_i \setminus C_i)$,
- $f_i(x) = 1$, if $x \in F_i$,
- $0 \leq f_i \leq 1$ on $\mathbb{R}^m$.

Put $f = c_1 \cdot f_1 + \cdots + c_n \cdot f_n$. This function satisfies claimed conditions since the sets $B_1, \ldots, B_n$ are pairwise disjoint.

Lemma 0.3.2 *Suppose that the set $A \subset \mathbb{R}^m$ is nonvoid, bounded and measurable, and function v is measurable, such that $\|v \cdot \chi_A\| \le c < \infty$. Then for each $\varepsilon > 0$ there exist approximately continuous functions f and g, such that*

i) $f(x) = g(x) = 0$ *for* $x \notin A$,

ii) $\|f\| \le c \vee \sqrt{c}$, $\|g\| \le 1 \wedge \sqrt{c}$,

iii) $\left|\int_I f(t)\,dt\right| \le \varepsilon$, $\left|\int_I g(t)\,dt\right| \le \varepsilon$ *for every interval* $I \in \mathcal{I}$,

iv) $\int_A |v(t) - f(t) \cdot g(t)|\,dt \le \varepsilon$.

Proof. Represent A as a union of nonempty, pairwise disjoint measurable sets $A_1, \ldots, A_n$, such that

$$\operatorname{diam} A_i \le \frac{\varepsilon \cdot (1 \vee \operatorname{diam} A)^{1-m}}{3m \cdot (1 \vee c)} \qquad \text{and} \qquad \omega(v, A_i) \le \frac{\varepsilon}{3|A| + 1}$$

for $i = 1, \ldots, n$. For $i = 1, \ldots, n$, find closed, pairwise disjoint sets $P_i, Q_i \prec A_i$ such that $|P_i| = |Q_i|$ and $|A_i \setminus (P_i \cup Q_i)| \le \varepsilon/(3n \cdot (1 \vee c))$. Choose also any $x_i \in A_i$. Put $a_i = |v(x_i)| \vee \sqrt{|v(x_i)|}$ and $b_i = \left(1 \wedge \sqrt{|v(x_i)|}\right) \cdot \operatorname{sgn}(v(x_i))$. Let f and g be Lebesgue functions such that:

- $f(x) = a_i$, $g(x) = b_i$, if $x \in P_i$, $i = 1, \ldots, n$,
- $f(x) = -a_i$, $g(x) = -b_i$, if $x \in Q_i$, $i = 1, \ldots, n$,
- $f(x) = 0$, $g(x) = 0$, if $x \notin A$,
- $\|f\| \le c \vee \sqrt{c}$, $\|g\| \le 1 \wedge \sqrt{c}$.

(We use Lemma 0.3.1.) Then i) and ii) are obviously satisfied. Since for $i = 1, \ldots, n$

$$\left|\int_{A_i} f(t)\,dt\right| = \left|\int_{A_i \setminus (P_i \cup Q_i)} f(t)\,dt\right| \le \frac{(c \vee \sqrt{c}) \cdot \varepsilon}{3n \cdot (1 \vee c)} \le \frac{\varepsilon}{3n},$$

so for every interval I

$$\begin{aligned}\left|\int_I f(t)\,dt\right| &= \left|\int_{I\cap A} f(t)\,dt\right| \\ &\le \sum_{i=1}^{n}\left|\int_{A_i} f(t)\,dt\right| + \sum_{A_i\setminus I\neq\emptyset}\left|\int_{I\cap A_i} f(t)\,dt\right| \\ &\le n\cdot\varepsilon/(3n) + \int_{I\cap\bigcup_{A_i\setminus I\neq\emptyset}A_i}|f(t)|\,dt \le \varepsilon/3 + \|f\|\cdot\left|I\cap\bigcup_{A_i\setminus I\neq\emptyset}A_i\right| \\ &\le \varepsilon/3 + \left(c\vee\sqrt{c}\right)\cdot 2m\cdot\max\{\operatorname{diam}A_i : i=1,\dots,n\}\cdot(\operatorname{diam}A)^{m-1} \\ &\le \varepsilon/3 + 2m\cdot\left(c\vee\sqrt{c}\right)\cdot(\operatorname{diam}A)^{m-1}\cdot\frac{\varepsilon\cdot(1\vee\operatorname{diam}A)^{1-m}}{3m\cdot(1\vee c)} \le \varepsilon.\end{aligned}$$

Similarly $\left|\int_{A_i} g(t)\,dt\right| \le \varepsilon/(3n)$ for $i=1,\dots,n$ and $\left|\int_I g(t)\,dt\right| \le \varepsilon$ for every interval I. Clearly

$$\begin{aligned}\int_A |v(t)-f(t)\cdot g(t)|\,dt &= \sum_{i=1}^{n}\int_{A_i}|v(t)-f(t)\cdot g(t)|\,dt \\ &\le \sum_{i=1}^{n}\int_{P_i\cup Q_i}|v(t)-f(t)\cdot g(t)|\,dt + \sum_{i=1}^{n}\int_{A_i\setminus(P_i\cup Q_i)}|v(t)-f(t)\cdot g(t)|\,dt \\ &\le \sum_{i=1}^{n}\|v-f(x_i)\cdot g(x_i)\|\cdot|A_i| + \sum_{i=1}^{n}\|v-f\cdot g\|\cdot|A_i\setminus(P_i\cup Q_i)| \\ &\le \sum_{i=1}^{n}\omega(v,A_i)\cdot|A_i| + \sum_{i=1}^{n}2c\cdot\frac{\varepsilon}{3n\cdot(1\vee c)} \le \frac{\varepsilon}{3|A|+1}\cdot|A| + 2n\cdot\frac{\varepsilon}{3n} \le \varepsilon,\end{aligned}$$

which completes the proof.

Lemma 0.3.3 *Assume that $H_1, H_2, \dots$ is a sequence of pairwise disjoint compact subsets of $\mathbb{R}^m$ and $K_1, K_2, \dots$ is a sequence of non-negative real numbers such that the function $\sum_{n=1}^{\infty} K_n\cdot\chi_{H_n}$ belongs to the first class of Baire. Then there is a sequence $\varepsilon_1, \varepsilon_2, \dots$ of positive numbers such that the following conditions hold:*

i) *if $f_1, f_2, \dots$ are derivatives, such that for each $n\in\mathbb{N}$, $|f_n| \le K_n\cdot\chi_{H_n}$ and $\left|\int_I f_n(t)\,dt\right| \le \varepsilon_n$ for every interval I, then function $f=\sum_{n=1}^{\infty} f_n$ is a derivative.*

ii) *if $w_1, w_2, \ldots$ are approximately continuous functions, such that for each $n \in \mathbb{N}$, $|w_n| \le K_n \cdot \chi_{H_n}$ and $\left|\int_{\mathbb{R}^m} w_n(t)\,dt\right| \le \varepsilon_n$, then function $w = \sum_{n=1}^{\infty} w_n$ is a Lebesgue function.*

Proof. Since the function $\sum_{n=1}^{\infty} K_n \cdot \chi_{H_n}$ is in the first class of Baire, there exists a family of compact sets $\mathcal{T} = \{T_i : i \in \mathbb{N}\}$, such that for each $r > 0$ the set $\{x \in \mathbb{R}^m : \sum_{n=1}^{\infty} K_n \cdot \chi_{H_n} < r\}$ is a union of some subfamily of $\mathcal{T}$. For each $n \in \mathbb{N}$ set $\widetilde{H}_n = \bigcup_{i<n} H_i \cup \bigcup_{\substack{T_i \cap H_n = \emptyset \\ i<n}} T_i$ and $\varepsilon_n = 2^{-n} \cdot \left(1 \wedge \left(\varrho\left(H_n, \widetilde{H}_n\right)\right)^{m+1}\right)$.

i) Since for every interval I the series $\sum_{n=1}^{\infty} \int_I f_n(t)\,dt$ is absolutely convergent, so the interval function $F(I) = \sum_{n=1}^{\infty} \int_I f_n(t)\,dt$ is additive. We will show that it satisfies the following condition:

(Σ) *for each $x \in \mathbb{R}^m$ and each $\varepsilon > 0$ there exists an $\eta > 0$, such that for every $I \in \mathcal{I}$, if $x \in I$ and* diam *$I < \eta$, then* $\dfrac{|F(I) - f(x) \cdot |I||}{(\operatorname{diam} I)^m} < \varepsilon$.

Then for each $x \in \mathbb{R}^m$ and each sequence $\{I_n : n \in \mathbb{N}\} \subset \mathcal{I}$ convergent to x we will have

$$\lim_{n\to\infty} \left|\frac{F(I_n)}{|I_n|} - f(x)\right| = \lim_{n\to\infty} \left|\frac{F(I_n) - f(x)\cdot|I_n|}{|I_n|}\right|$$
$$\le \lim_{n\to\infty} \left|\frac{F(I_n) - f(x)\cdot|I_n|}{(\operatorname{diam} I)^m}\right| \cdot \limsup_{n\to\infty} \frac{(\operatorname{diam} I)^m}{|I_n|} = 0,$$

i.e., $\lim_{I \Rightarrow x} \dfrac{F(I)}{|I|} = f(x)$.

Take an $x \in \mathbb{R}^m$ and $\varepsilon > 0$. Note first that if $x \in H_p$ for some $p \in \mathbb{N}$, then for every $n > p$ and every interval I, if $x \in I$, then:

- $H_n \cap I \ne \emptyset$ implies $\varrho\left(H_n, \widetilde{H}_n\right) \le \operatorname{diam} I$, so
$$\left|\int_I f_n(t)\,dt\right| \le \varepsilon_n \le 2^{-n} \cdot \left(\varrho\left(H_n, \widetilde{H}_n\right)\right)^{m+1} \le 2^{-n} \cdot (\operatorname{diam} I)^{m+1},$$
- $H_n \cap I = \emptyset$ implies $\int_I f_n(t)\,dt = 0$.

Hence if $\operatorname{diam} I < \varepsilon$, then $\left|\dfrac{\sum_{n=p+1}^{\infty} \int_I f_n(t)\,dt}{(\operatorname{diam} I)^m}\right| \le \varepsilon$, i.e., $\sum_{n=p+1}^{\infty} \int_I f_n(t)\,dt$ satisfies condition (Σ) with respect to $\sum_{n=p+1}^{\infty} f_n$, and since each of functions $f_1, \dots, f_p$ is a derivative, so condition (Σ) is in this case satisfied.

On the other side, if $x \notin \sum_{n=1}^{\infty} H_n$, then for $r = \varepsilon/2$ choose $p \in \mathbb{N}$, such that $x \in T_p$ and $\sum_{n=1}^{\infty} K_n \cdot \chi_{H_n} < r$ on T_p. Then for every $n > p$ and every interval I, if $x \in I$, then:

- $H_n \cap T_p \neq \emptyset$ implies $K_n < r$, so
$$\left|\int_I f_n(t)\,dt\right| \le r \cdot |I| \le r \cdot (\operatorname{diam} I)^m,$$
- if $H_n \cap T_p = \emptyset$ and $H_n \cap I \neq \emptyset$, then $\varrho\left(H_n, \widetilde{H}_n\right) \le \operatorname{diam} I$, so
$$\left|\int_I f_n(t)\,dt\right| \le \varepsilon \le 2^{-n} \cdot \left(\varrho\left(H_n, \widetilde{H}_n\right)\right)^{m+1} \le 2^{-n} \cdot (\operatorname{diam} I)^{m+1},$$
- finally $H_n \cap I = \emptyset$ implies $\int_I f_n(t)\,dt = 0$.

Hence for every interval I of diameter less than $\varrho(x, \bigcup_{n=1}^{p} H_n) \wedge (\varepsilon/2)$
$$\left|\frac{\sum_{n=1}^{\infty} \int_I f_n(t)\,dt}{(\operatorname{diam} I)^m}\right| \le r + \operatorname{diam} I \le \varepsilon,$$
so condition (Σ) is in this case also satisfied, because $f(x) = 0$.

ii) Let $x \in \mathbb{R}^m$. Put $u = \sum_{H_n \ni x} w_n$ and $v = w - u$. Since u is bounded and approximately continuous, x is a Lebesgue point for u. On the other side from i) $|v|$ is a derivative, and since $v(x) = 0$, x is a Lebesgue point for v. Hence x is a Lebesgue point for w.

Lemma 0.3.4 *Whenever u is a function of the first class of Baire there exist a function of the first class of Baire v, a sequence $\{H_n : n \in \mathbb{N}\}$ of pairwise disjoint compact sets and a sequence $\{c_n : n \in \mathbb{N}\}$ of positive numbers such that the following conditions are satisfied:*

i) *$u - v$ is an approximately continuous function.*

ii) *v is approximately continuous at all points of $\bigcup_{n=1}^{\infty} H_n$.*

iii) *$v(x) = 0$, if $x \in H_n$ for some $n \in \mathbb{N}$ and x is not a density point of H_n.*

iv) *$|v| \le \sum_{n=1}^{\infty} c_n \cdot \chi_{H_n}$ on $\mathbb{R}^m$.*

v) *$\sum_{n=1}^{\infty} c_n \cdot \chi_{H_n}$ belongs to the first class of Baire.*

vi) *v is bounded provided that u is bounded.*

Proof. Let E be a set of measure 0 containing all points of approximate discontinuity of u, such that $\mathbb{R}^m \setminus E$ is a 0-dimensional space. Let φ be an approximately continuous function, such that $\varphi(x) = u(x)$ for $x \in E$ (Theorem 0.2.3). Put $v_1 = u - \varphi$ and $X = \{x \in \mathbb{R}^m : \varphi(x) \ne u(x)\}$. Then function $\log|v_1|$ is a function of the first class of Baire on X, so there exists a function $g : X \longrightarrow \mathbb{R}$ of the first class of Baire on X, such that the set of its values is discrete and $|\log|v_1(x)| - g(x)| \le 1$ for $x \in X$ (§31, Chapter VIII, Theorem 3 in [3]). Using that X is a 0-dimensional space and an F_σ set, we can represent X as the union of pairwise disjoint compact sets $X = \bigcup_{n=1}^{\infty} H_n$, such that g is constant on each H_n (§30, Chapter V in [3]). Put $w(x) = \exp(g(x) + 1)$, if $x \in X$ and $w(x) = 0$ otherwise. Let c_n be the value of w on H_n ($n \in \mathbb{N}$). Then condition v) is satisfied because w is non-negative and for every $a > 0$

- $\{x \in \mathbb{R}^m : w(x) > a\} = \{x \in X : g(x) > -1 + \log a\} \in F_\sigma$,
- $\{x \in \mathbb{R}^m : w(x) < a\} =$
$$= \{x \in X : g(x) < -1 + \log a\} \cup \{x \in \mathbb{R}^m : |v_1(x)| < a \cdot e^{-2}\} \in F_\sigma.$$

Let E_1 be a set of measure 0 containing all points of approximate discontinuity of function v_1 and all $x \in \mathbb{R}^m$, such that $x \in H_n$ for some $n \in \mathbb{N}$ and x is not a density point of H_n. Put $\psi = (\psi_1 \wedge (v_1 \vee 0)) \vee (v_1 \wedge 0)$, where ψ_1 is an approximately continuous function, such that $\psi_1(x) = u(x)$ for $x \in E_1$ (Theorem 0.2.3). Then function ψ is clearly approximately continuous, and since $v_1 \wedge 0 \le \psi \le v_1 \vee 0$ and $|v_1| \le w = \sum_{n=1}^{\infty} c_n \cdot \chi_{H_n}$, so $v = v_1 - \psi$ satisfies condition iv). The other conditions are obvious.

Theorem 0.3.5 *Whenever u is a function of the first class there exist a derivative f, a bounded derivative g and an aproximately continuous function h such that $u = f \cdot g + h$ (all notions with respect to the ordinary differentiation basis). In case u is bounded we can find such a representation that f and h are also bounded. (So, in particular, h is a Lebesgue function.)*

Proof. Choose an approximately continuous function v a sequence of pairwise disjoint compact sets $\{H_n : n \in \mathrm{N}\}$ and a sequence of reals $\{c_n : n \in \mathrm{N}\}$ according to Lemma 0.3.4. For the sequences $\{H_n : n \in \mathrm{N}\}$ and $\{K_n : n \in \mathrm{N}\}$, where $K_n = c_n \vee \sqrt{c_n}$, we find positive numbers $\{\varepsilon_n : n \in \mathrm{N}\}$ according to Lemma 0.3.3. For each n use Lemma 0.3.2 with $A = H_n$ and $\varepsilon = \varepsilon_n$ to construct functions f_n and g_n with properties described there. From Lemma 0.3.3 we see that the functions $f = \sum_{n=1}^{\infty} f_n$ and $g = \sum_{n=1}^{\infty} g_n$ are derivatives. Using the conditions ii) and iii) of Lemma 0.3.4 we get that for each $n \in \mathrm{N}$ function $v \cdot \chi_{H_n}$ is approximately continuous. By Lemma 0.3.3 we get that function $v - f \cdot g$ is a Lebesgue function (since $v - f \cdot g = \sum_{n=1}^{\infty} (v \cdot \chi_{H_n} - f_n \cdot g_n)$). Therefore the function $h = u - f \cdot g = (u - v) + (v - f \cdot g)$ is approximately continuous.

If u is bounded then so is v and, consequently, so is f and hence so is h. The other conditions are obvious.

0.4 Queries.

Query 0.4.1* *Given an unbounded Baire one function $u : \mathbb{R}^m \longrightarrow \mathbb{R}$, can we find derivatives f, g and h (with respect to the ordinary differentiation basis) such that $u = f \cdot g + h$?*

Query 0.4.2 *Given a function u in the first class of Baire, can we find derivatives f, g and h (with respect to the strong differentiation basis) such that $u = f \cdot g + h$?*

(We recall that strong differentiation basis is a pair $(\mathcal{I}, \Rightarrow)$, such that $\mathcal{I}$ is the family of all m-dimensional intervals and $\{I_n : n \in \mathrm{N}\} \Rightarrow x$ iff $\{x\} = \bigcap_{n=1}^{\infty} I_n$

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*Recently, R. Carrese answered this question in the positive—R. Carrese, *On the algebra generated by derivatives of interval functions*, Real Analysis Exchange 14 (2), (1988–89), 307–320.

and $\lim_{n\to\infty} \operatorname{diam} I_n = 0$.)

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