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On Borel classes of multifunctions of two variables

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The content of the work announced during the Summer School on Real Functions 1988 is presented. It concerns a generalization of theorems on $\mathcal{B}$-measurability of functions of two variables (like the Marczewski-Nardzewski's and Kempisty's theorems) to the case of multifunctions. The theorems are given without proofs, which can be found in [3].

Let $(T, \mathcal{T})$ denote a topological space, $\mathcal{P}(T)$ the family of all nonempty subsets of T and $\mathcal{K}(T)$ the family of all compact subsets of T.

By $\Sigma_\alpha(T, \mathcal{T})$ and $\Pi_\alpha(T, \mathcal{T})$ we denote respectively the additive and multiplicative classes α in Borel hierarchy of subsets of T. Finally let $(Z, \mathcal{T}_Z)$ denote a topological space.

Definition 1 *A multifunction $F : T \longrightarrow \mathcal{P}(Z)$ is said to be in lower (resp. upper) class α if for every $\mathcal{T}_Z$-open set G the set $F^-(G) = \{t \in T : F(t) \cap G \neq \emptyset\}$ is in $\Sigma_\alpha(T, \mathcal{T})$ (resp. if the set $F^+(G) = \{t \in T : F(t) \subset G\}$ is in $\Sigma_\alpha(T, \mathcal{T})$).*

A multifunction which is in lower (resp. upper) class 0 is called $\mathcal{T}$-lower (resp. $\mathcal{T}$-upper) semicontinuous. A multifunction which is simultaneously $\mathcal{T}$-lower and $\mathcal{T}$-upper semicontinuous is called $\mathcal{T}$-continuous.

Let X and Y denote arbitrary spaces. For a multifunction $F : X \times Y \longrightarrow \mathcal{P}(Z)$ the multifunction $F_x : Y \longrightarrow \mathcal{P}(Z)$ such that $F_x(y) = F(x, y)$ (resp. $F^y : X \longrightarrow \mathcal{P}(Z)$ such that $F^y(x) = F(x, y)$) denotes x-section (resp. y-section) of F.

Theorem 2 *([3], theorem 2) Let $(X, \mathcal{T}_X$ and $(Z, \mathcal{T}_Z)$ denote two perfectly normal topological spaces and let (Y, d) denote a metric space. Let $\mathcal{T}_Y$ be a topology in Y finer than metric one such that $(T, \mathcal{T}_Y)$ is separable. Let S be some fixed countable and dense subset of Y. Suppose that for every $v \in Y$ there corresponds a subset $U(v) \in \mathcal{T}_Y$ such that*

1. $\forall_{y \in S}\ B_y = \{v : y \in U(v)\} \in \Sigma_\alpha(Y, d)$,
2. *the family $\mathcal{N}(v) = \{U(v) \cap K(v, 2^{-n}) : n = 1, 2, \dots\}$ (where $K(v, 2^{-n})$ denotes the open ball with center at v and radius 2^{-n}) forms a filterbase of $\mathcal{T}_Y$-neighbourhoods of point v.*

Assume that $F : X \times Y \longrightarrow \mathcal{P}(Z)$ is a multifunction whose all y-sections are in upper class α and all x-sections are $\mathcal{T}_Y$-continuous. Then F belongs to the lower class $\alpha + 1$ in the product $(X, \mathcal{T}_X) \times (Y, d)$.

The next theorem 3 may be viewed as a sort of dualization of theorem 2.

Theorem 3 *([3], theorem 5) Let X, Y and Z be the same as in theorem 2. Let $F : X \times Y \longrightarrow \mathcal{K}(Z)$ be a multifunction whose all y-sections are in lower class α and all x-sections are $\mathcal{T}_Y$-continuous. Then F belongs to the upper class $\alpha + 1$ in $(X, \mathcal{T}_X) \times (Y, d)$.*

Conditions imposed on Y are some generalization of right-continuity contained in papers of Dravecky-Neubrunn [1] and Marczewski-Nardzewski [4].

Theorem 4 *([3], theorem 7) Let $(X, \mathcal{T}_X)$ be a topological space, (Y, d) a metric space and $(Z, \mathcal{T}_Z)$ a perfectly normal topological space. Let $F : X \times Y \longrightarrow \mathcal{K}(Z)$ be a multifunction whose all y-sections are $\mathcal{T}_X$-lower semicontinuous and all x-sections are $\mathcal{T}_Y$-upper semicontinuous (where $\mathcal{T}_Y$ denotes the topology in Y generated by metric d). Then F belongs to the upper class 1 in $(X, \mathcal{T}_X) \times (Y, d)$.*

Theorem 5 *([3], theorem 10) Let Y and X be the same as in theorem 4 and let (Z, ϱ) be a separable metric space. Let $F : X \times Y \longrightarrow \mathcal{P}(Z)$ be a multifunction whose all y-sections are $\mathcal{T}_X$-lower semicontinuous and all x-sections are $\mathcal{T}_Y$-upper semicontinuous. Then F belongs to the lower class 1 in $(X, \mathcal{T}_X) \times (Y, d)$.*

There exists an example showing that the metrizability of one of the spaces X and Y in theorems 4 and 5 is essential.

Theorem 5 gives a solution of the problem 2 raised by Wł. Ślęzak in his paper [5] on p. 88.

From theorems 4 and 5 we conclude that if $F : X \times Y \longrightarrow \mathcal{K}(Z)$ is a multifunction with $\mathcal{T}_X$-lower semicontinuous y-sections and $\mathcal{T}_Y$-upper semicontinuous x-sections, then F is in the first class of Baire as a single valued function into the hyperspace $\mathcal{K}(Z)$ endowed with exponential topology.

If both the x-sections and the y-sections of a multifunction F are lower or upper semicontinuous, then its behaviour may be very bad. To be more specific it is possible to show that

Theorem 6 *([3], example 13) There exists a multifunction $F : \Re \times \Re \longrightarrow \mathcal{K}(\Re)$ whose all x-sections and y-sections are lower semicontinuous and upper semicontinuous except one point belonging to neither lower nor upper classes.*

It can be proved that

Theorem 7 *If the multifunction $F : X \times Y \longrightarrow \mathcal{P}(Z)$, where $(X, \mathcal{T}_X)$ is perfectly normal topological space, (Y, d) and (Z, ϱ) are separable metric spaces, has all x-sections lower semicontinuous and upper quasicontinuous and all y-sections lower semicontinuous, then F is in lower class 2 and this class is the best possible.*

References.

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