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On the strong semi-continuity of functions

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We introduce a definition analogous to the Grande's definition contained in [2].

Let (X, d) be a metric space.

Definition 1 *A function $F : X \longrightarrow \Re$ is called strongly upper semi-continuous at a point $x_0 \in X$ if it is upper semi-continuous at this point and there exists an open set $U \subset X$ such that $x_0 \in \mathrm{Cl}\,U$ and $\lim_{x \to x_0, x \in U} f(x) = f(x_0)$ (where* Cl *denotes the closure operator).*

A function that is strongly upper semi-continuous at any point $x_0 \in X$ is called strongly upper semi-continuous.

Analogously the strong lower semi-continuity of function f can be defined.

Observe that sum, product and minimum of two strongly upper semi-continuous functions need not be strongly upper semi-continuous. As the example it is sufficient to take two following functions:

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases} \quad \text{and } g(x) = \begin{cases} 2 & \text{if} \quad x \leq 0 \\ \frac{1}{2} & \text{if} \quad x > 0 \end{cases}$$

The strong semi-continuity can be characterized with the aid of quasi-continuity.

Definition 2 *A function $f : X \longrightarrow \Re$ is called quasi-continuous (resp. lower quasi-continuous) at a point $x_0 \in X$ if for every number $\varepsilon > 0$ and every neighbourhood $W(x_0)$ of point x_0 there exists a nonempty and open set $V \subset W(x_0)$ such that $|f(x) - f(x_0)| < \varepsilon$ (resp. $f(x_0) - f(x) < \varepsilon$) for every point $x \in V$.*

A set $A \subset X$ is called semi-open if there exists an open set $G \subset X$ such that $G \subset A \subset \mathrm{Cl}\, G$. Evidently every open set is also semi-open. Intersection $A \cap V$ of semi-open set A and open set V is semi-open.

A function $F : X \longrightarrow \Re$ is lower quasi-continuous at a point $x_0 \in X$ iff for every number $\varepsilon > 0$ there exists semi-open neighbourhood $A(x_0)$ of x_0 such that $A(x_0) \subset \{x : f(x_0) - f(x) < \varepsilon\}$ (see [1]).

Theorem 3 *Suppose that $F : X \longrightarrow \Re$ is an upper semi-continuous function. Then the following conditions are equivalent:*

1. *f is strongly upper semi-continuous,*
2. *f is quasi-continuous,*
3. *f is lower quasi-continuous.*

Proof. $1 \Rightarrow 2$. Fix a point $x_0 \in X$. Let $\varepsilon > 0$ and $W(x_0)$ be arbitrary. The function f is strongly upper semi-continuous at x_0 therefore there exists an open set $U \subset X$ such that $x_0 \in \mathrm{Cl}\, U$ and $\lim_{x \to x_0, x \in U} f(x) = f(x_0)$. Then there exists an open neighbourhood $V(x_0)$ of x_0 such that $V(x_0) \cap U \subset \{x \in X : |f(x_0) - f(x)| < \varepsilon\}$. Let $V = W(x_0) \cap V(x_0) \cap U$. Evidently V is an open subset of $W(x_0)$ and $V \neq \emptyset$ because $x_0 \in \mathrm{Cl}\, U$. Now f is quasi-continuous at x_0.

$2 \Rightarrow 3$ is evident.

$3 \Rightarrow 1$. Fix $\varepsilon > 0$. The function f is lower quasi-continuous at x_0. Then there exists a semi-open neighbourhood $A(x_0)$ of x_0 such that

$$A(x_0) \subset \{x \in X : f(x_0) - f(x) < \varepsilon\}. \tag{1}$$

The function f is also upper semi-continuous at x_0, so there exists an open neighbourhood $V(x_0)$ of x_0 such that

$$V(x_0) \subset \{x \in X : f(x) - f(x_0) < \varepsilon\}. \tag{2}$$

Evidently the set $A(x_0) \cap V(x_0)$ is nonempty and semi-open. Let $U = \text{Int}\,(A(x_0) \cap V(x_0))$. Now by 1 and 2 f is strongly upper semi-continuous and the proof of the theorem is completed.

Let X and Y be arbitrary spaces and let $f : X \times Y \longrightarrow \Re$ be a function. Then the function $f_x : Y \longrightarrow \Re$ for $x \in X$ (resp. $f^y : X \longrightarrow \Re$ for $y \in Y$) such that $f_x(y) = f(x,y)$ (resp. $f^y(x) = f(x,y)$) is called as usually x-section (resp. y-section) of f.

Let (X,d) and (Y,ϱ) be metric spaces. If a function $f : X \times Y \longrightarrow \Re$ is upper semi-continuous then all x-sections and all y-sections of f are obviously upper semi-continuous.

Let $T \subset \Re^2$ denote a closed triangle on the plain and let

$$f(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in T = Conv\{(0,0),(0,1),(1,0)\} \\ 0 & \text{if} \quad (x,y) \notin T \end{cases} .$$

Now it is easy to see that if a function $f : X \times Y \longrightarrow \Re$ is strongly upper semi-continuous then its x-sections and y-sections need not be strongly upper semi-continuous.

For the proof of next theorem we quote Ślęzak's theorem from [4].

Let $(X, \mathcal{T}_X)$ and $(Z, \mathcal{T}_Z)$ denote two topological spaces and let $F : X \longrightarrow Z$ denote a multifunction. Let $F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}$ and $F^+(G) = \{x \in X : F(x) \subset G\}$. It is easy to see that

$$F^+(G) = X \setminus F^-(Z - G). \tag{3}$$

Let $\Sigma_\alpha(X)$ and $\Pi_\alpha(X)$ denote respectively additive and multiplicative class α, $\alpha < \Omega$, in Borel hierarchy of subsets of X, i.e. $\Sigma_0(X)$ and $\Pi_0(X)$ denote respectively the family of open and closed subsets of X, $\Sigma_1(X) = F_\sigma$ and $\Pi_1(X) = G_\delta$, $\Sigma_2(X) = G_{\delta\sigma}$ and $\Pi_2(X) = F_{\sigma\delta}, \ldots$.

Theorem 4 *([4], theorem 1) Let $(X, \mathcal{T}_X)$ be a perfectly normal topological space and let Z be a Polish space. Suppose that $F : X \longrightarrow Z$ is a multifunction with closed values. Then the following conditions are equivalent:*

1. *F is of lower class α ($\alpha > 0$), i.e. $F^-(G) \in \Sigma_\alpha(X)$ for every open set $G \subset Z$,*

2. *there exist Borel α functions $f_n : X \longrightarrow Z$, $n = 1,2,\ldots$ such that for every $x \in X$ we have $F(x) = \text{Cl}\,\{f_n(x) : n = 1,2,\ldots\}$.*

Theorem 5 *Let (X,d) be a metric space and let (Y,ϱ) be a separable and complete metric space. Let $f : X \times Y \longrightarrow \Re$ be a function such that all x-sections are strongly upper semi-continuous and all y-sections are upper semi-continuous. Then the function f belongs to the upper class 2 in the Young classification, i.e. $f^{-1}(-\infty, r) \in G_{\delta\sigma}$ for every $r \in \Re$.*

Proof. Let $S = \{s_1, s_2, \ldots, s_n, \ldots\}$ be an arbitrary countable ϱ-dense subset of Y. Since all x-sections of f are strongly upper semi-continuous then there exists an open set $U \subset Y$ such that

$$y \in \mathrm{Cl}\, U \text{ and } \lim_{z \to y, z \in U} f(x,z) = f(x,y).$$

Therefore to each point $(x,y) \in X \times Y$ there corresponds a sequence $n \mapsto s_n(x,y) \in S$ such that

$$\lim_{n\to\infty} s_n(x,y) = (x,y) \text{ and } \lim_{n\to\infty} f(x, s_n(x,y)) = f(x,y). \tag{4}$$

Let $Q = \{q_1, q_2, \ldots, q_m, \ldots\}$ be an enumeration of the rational numbers. For every $(n,m) \in \mathcal{N} \times \mathcal{N}$ define a complex function $f_{nm} : X \longrightarrow Y \times \Re$ by formula

$$f_{nm}(x) = (s_n, \min(q_m, f(x, s_n))) \text{ for } x \in X. \tag{5}$$

Clearly

$$\text{all } f_{nm} \text{ are Borel class 1 functions} \tag{6}$$

because $f^{s_n} : X \longrightarrow Y \times \Re$ are upper semi-continuous functions and then of Borel class 1.

Let $H(x) = \{f_{nm}(x) : (n,m) \in N \times N\}$ for $x \in X$.

Define a multifunction $F : X \longrightarrow Y \times \Re$ by formula

$$F(x) = \{(y,r) \in Y \times \Re : f(x,y) \geq r\} \text{ for } x \in X.$$

Notice that

$$\text{for every } x \in X,\ F(x) \text{ is a closed subset of } Y \times \Re \tag{7}$$

by virtue of the upper semi-continuity of x-sections.

It is easy to show that

(8) for every $x \in X$, $F(x) = \mathrm{Cl}\, H(x)$.

Indeed, let $(y,r) \in F(x)$. Then $f(x,y) \geq r$. For the point (x,y) there exists a sequence $s_n(x,y) \in S$ such that 4 holds. Analogously for the number $r \in \Re$ there exists a sequence $q_m \in Q$ such that $\lim_{n\to\infty} q_m = r$ and $q_m \leq f(x, s_n(x,y))$. Therefore

$$\lim_{n\to\infty}\lim_{m\to\infty} f_{nm}(x) = \lim_{n\to\infty}\lim_{m\to\infty}(s_n, \min(q_m, f(x, s_n))) = (y,r)$$

and accordingly $(y,r) \in \mathrm{Cl}\, H(x)$.

On the other hand $H(x) \subset F(x)$. Then $\mathrm{Cl}\, H(x) \subset \mathrm{Cl}\, F(x) = F(x)$ and 8 is true.

According to 6, 7 and 8, by theorem 4 we have that the multifunction F is in lower class 1, i.e.

(9) $F^-(G) \in F_\sigma$ for every open set $G \subset Y \times \Re$.

Let $\mathcal{G}r(F) = \{(x,y,r) \in X \times Y \times \Re : (y,r) \in F(x)\}$ denote the graph of F. Observe that

(10) $\mathcal{G}r(F) \in F_{\sigma\delta}$.

Indeed, let $G_1, G_2, \ldots$ be a countable open base in the product space $Y \times \Re$. If $(y,r) \notin F(x)$ then there exists an $n \in \mathcal{N}$ such that $(y,r) \in G_n$ and $F(x) \cap G_n = \emptyset$. Therefore we have

$$\begin{aligned} X \times (Y \times \Re) \setminus \mathcal{G}r(F) &= \bigcup_{n=1}^{\infty} [\{x \in X : F(x) \cap G_n = \emptyset\} \times G_n] \\ &= \bigcup_{n=1}^{\infty} [\{x \in X : F(x) \subset (Y \times \Re) \setminus G_n\} \times G_n] \\ &= \bigcup_{n=1}^{\infty} \left[F^+((Y \times \Re) \setminus G_n) \times G_n\right]. \end{aligned}$$

Moreover $F^+((Y \times \Re) \setminus G_n) = X \setminus F^-(G_n)$ (see 3). Then by 9 $F^+((Y \times \Re) \setminus G_n) \in G_{\delta\sigma}$ and $F^+(((Y \times \Re) \setminus G_n) \times G_n) \in G_{\delta\sigma}$. Accordingly $X \times (Y \times \Re) \setminus \mathcal{G}r(F) \in G_{\delta\sigma}$ and 10 is true.

Moreover for every $r \in \Re$, r-section of the set $\mathcal{G}r(F)$, i.e. the set

$$
\begin{aligned}
(\mathcal{G}r(F))^r &= \{(x,y) \in X \times Y : (x,y,r) \in \mathcal{G}r(F)\} \\
&= \{(x,y) \in X \times Y : (y,r) \in F(x)\} \\
&= \{(x,y) : f(x,y) \geq r\} \in F_{\sigma\delta}. \qquad (11)
\end{aligned}
$$

Let $r \in \Re$ be an arbitrary real number. Now we have

$$
\begin{aligned}
f^{-1}(-\infty, r) &= \{(x,y) \in X \times Y : f(x,y) < r\} \\
&= X \times Y \setminus \{(x,y) \in X \times Y : f(x,y) \geq r\}.
\end{aligned}
$$

Therefore by 10 we have $f^{-1}(-\infty, r) \in G_{\delta\sigma}$ and proof of the theorem is completed.

The theorem mentioned above is a generalization of theorem 5 in [2] and showing moreover that the function f is in lower class 2. The measurability of function f we can obtain after weakening of assumption about y-sections of f. Remark moreover that the proof given here differs from Grande's one and maybe it is more direct.

Before the explanation of some details let us make known the next theorem.

Theorem 6 *([3], theorem 5.6) Let $(X, \mathcal{X})$ be a measurable space and let (Z, d) be a separable metric space. Let $F : X \longrightarrow Z$ be a multifunction with complete values. Then the following conditions are equivalent:*

1. *F is weakly measurable, i.e. $F^{-}(G) \in \mathcal{X}$ for every open set $G \subset Z$,*

2. *there exist $\mathcal{X}$-measurable functions $f_n : X \longrightarrow Z$, $n = 1, 2, \ldots$ such that for every $x \in X$ we have $F(x) = \mathrm{Cl}\{f_n(x) : n = 1, 2, \ldots\}$.*

Theorem 7 *Let $(X, \mathcal{X}, d)$ be a measurable metric space. Let (Y, ϱ) be a separable and complete metric space. Let $f : X \times Y \longrightarrow \Re$ be a function such that all its x-sections are strongly upper semi-continuous and all its y-sections are $\mathcal{X}$-measurable. Then the function f is $\mathcal{X} \times \mathcal{B}(Y)$-measurable.*

Proof. Define, analogously as in the proof of theorem 5, the functions $f_{nm} : X \longrightarrow Y \times \Re$ for $(n,m) \in \mathcal{N} \times \mathcal{N}$. Evidently

(12) for every $(n,m) \in \mathcal{N} \times \mathcal{N}$ the function f_{nm} is $\mathcal{X}$-measurable.

Analogously let $H(x) = \{f_{nm}(x) : (n,m) \in \mathcal{N} \times \mathcal{N}\}$ and $F(x) = \{(y,r) \in Y \times \Re : f(x,y) \geq r\}$ for $x \in X$. We have again

(13) $$\text{for every } x \in X,\ F(x) = \operatorname{Cl} H(x).$$

From 12 and 13, by theorem 6, the multifunction F is weakly measurable. Thus the graph $\mathcal{G}r(F)$ of F is $\mathcal{X} \times \mathcal{B}(Y) \times \mathcal{B}(\Re)$-measurable (compare theorem 3.5 in [3]) and the r-section $(\mathcal{G}r(F))^r$ of $Gr(F)$ is $\mathcal{X} \times \mathcal{B}(Y)$-measurable for fixed $r \in \Re$. Therefore we have $f^{-1}(-\infty, r) = \{(x,y) \in X \times Y : f(x,y) < r\} = X \times Y \setminus (\mathcal{G}r(F))^r \in \mathcal{X} \times \mathcal{B}(Y)$ and proof of the theorem is completed.

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