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On $\mathcal{D}^{**}$-Darboux functions

Tomasz Natkaniec

Let us establish some of the terminology to be used. $\Re$ denotes the real line and $\mathcal{N}$ denotes the set of all positive integers. Let $A \in \Re$ be a c-dense in itself set and let B be a subset of $\Re$. We say that $f : A \longrightarrow B$ is an (A, B)-Darboux function iff f has the intermediate value property, i.e. $(f(x), f(y)) \cap B \subset f((x, y) \cap A)$ for each $x, y \in A$. Let $\mathcal{D}(A, B)$ denote the class of all (A, B)-Darboux functions and let $\mathcal{D}^{**}(A, B)$ be the class of all functions $f : A \longrightarrow B$ which take on every $y \in B$ c times in every non-empty set of the form $I \cap A$, where I is an interval (c denotes the cardinality of the continuum). It is clear that $\mathcal{D}^{**}(A, B) \subset \mathcal{D}(A, B)$ for each bilaterally c-dense subset A of $\Re$ and every subset B of $\Re$. For $A = B = \Re$ we shall denote the classes $\mathcal{D}(A, B)$ and $\mathcal{D}^{**}(A, B)$ by $\mathcal{D}$ and $\mathcal{D}^{**}$.

Let us remark that the class $\mathcal{D}$ is equal to the family of all Darboux functions and the class $\mathcal{D}^{**}$ is equal to the family of all Darboux functions for which all level sets are c-dense in $\Re$. These classes are well-known and studied by many mathematicians (see e.g. [1], [2] and [3]). If $B \neq \Re$ then the classes $\mathcal{D}(A, B)$ and $\mathcal{D}^{**}(A, B)$ are more special, nevertheless they are helpful in a discussion on many questions connected with the Darboux property.

For a family $\mathcal{A}$ of real functions let

$$\mathcal{M}_a(\mathcal{A}) = \{f : \forall_{g \in \mathcal{A}}\ f + g \in \mathcal{A}\},$$

$$\mathcal{M}_m(\mathcal{A}) = \{f : \forall_{g \in \mathcal{A}}\ fg \in \mathcal{A}\},$$

$$\mathcal{M}_{\max}(\mathcal{A}) = \{f : \forall_{g \in \mathcal{A}}\ \max(f, g) \in \mathcal{A}\}.$$

The classes $\mathcal{M}_a(\mathcal{D}^{**})$ and $\mathcal{M}_m(\mathcal{D}^{**})$ are characterized in [2] and [3]. In this note we shall prove that $\mathcal{M}_{\max}(\mathcal{D}^{**})$ is empty.

Theorem 1 *Let A, B, C be subsets of $\Re$, $F : A \times B \longrightarrow \Re$ and $f : \Re \longrightarrow A$. Then there exists a $d \in \mathcal{D}^{**}(\Re, B)$ such that $F(f,d) \in \mathcal{D}^{**}(\Re, C)$ iff the following conditions hold*

1. *for every $x \in \Re$ there exists $y \in B$ such that $F(f(x),y) \in C$,*

2. card $(\{x \in I : F(f(x),y) = c$ for some $y \in B\}) = c$ *for every $c \in C$ and every interval I,*

3. card $(\{x \in I : F(f(x),y) \in C\}) = c$ *for every $y \in B$ and every interval I.*

Proof. ($\Leftarrow$) Let $\mathcal{I}$ be the family of all non-empty open intervals in $\Re$. Well-order the set $\mathcal{I} \times B \times C$ as $(I_\alpha, y_\alpha, c_\alpha), \alpha < c$. We can choose (inductively) sequences x_α, z_α, t_α such that $x_\alpha, z_\alpha \in I_\alpha \setminus \{x_\beta, z_\beta : \beta < \alpha\}$, $x_\alpha \neq z_\alpha$, $F(f(x_\alpha), y_\alpha) \in C$, $t_\alpha \in B$ and $F(f(z_\alpha), t_\alpha) = c_\alpha$. Let us put

$$d(x) = \begin{cases} y_\alpha & \text{for } x = x_\alpha,\ \alpha < c, \\ t_\alpha & \text{for } x = z_\alpha,\ \alpha < c, \\ y \in B \text{ such that } F(f(x),y) \in C & \text{for } x \notin \{x_\alpha, z_\alpha : \alpha < c\}. \end{cases}$$

Then $d \in \mathcal{D}^{**}(\Re, B)$ and $F(f,d) \in \mathcal{D}^{**}(\Re, C)$.

($\Rightarrow$) Let $d \in \mathcal{D}^{**}(\Re, B)$ and $F(f,d) \in \mathcal{D}^{**}(\Re, C)$. Then 1 holds. Now let I be an interval and $c \in C$. Then card $(\{x \in I : F(f(x), d(x)) = c\}) = c$ and $d(x) \in B$, therefore

$$\text{card}\,(\{x \in I : F(f(x),y) = c \text{ for some } y \in B\}) = c.$$

Thus the condition 2 holds. Let $y \in B$. Since $d \in \mathcal{D}^{**}(\Re, B)$, card $(\{x \in I : d(x) = y\}) = c$. From $F(f(x), d(x)) \in C$ we obtain $\{x \in I : d(x) = y\} = \{x \in I : d(x) = y$ and $F(f(x), d(x)) \in C\} \subset \{x \in I : F(f(x), y) \in C\}$ and 3 is proved.

Remarks From Theorem 1 we can immediately obtain the following results.

1. For $A = B = \Re$, $F(x,y) = x + y$ we obtain Theorem 3 from [1], i.e. for $f : \Re \longrightarrow \Re$ there exists a $d \in \mathcal{D}^{**}$ such that $(f + d) \in \mathcal{D}^{**}(\Re, C)$ iff card $(\{x \in I : f(x) + y \in C\}) = c$ for every $y \in \Re$ and every interval I.

2. If $A = B = \Re$, $F(x,y) = x \cdot y$, $f : \Re \longrightarrow \Re$ and $0 \in C$ then we have the following corollary. For $f : \Re \longrightarrow \Re$ there exists a $d \in \mathcal{D}^{**}(\Re, C)$ iff card $(\{x \in I : f(x) \neq 0\}) = c$ for every interval I and card $(\{x \in I : f(x) \cdot y \in c\}) = c$ for every $y \in \Re$ and every interval I.

   Some interesting consequences of Remarks 1 and 2 we can find in [2] and [3], respectively.

3. Let $C = \{-1, 0, 1\}$, $A = \Re$, $B = (0, \infty)$ and $F(x,y) = x/y$. For every $f \in \mathcal{D}^{**}$ there exists a function $d \in \mathcal{D}^{**}(\Re, B)$ such that $f/d \in \mathcal{D}^{**}(\Re, C)$ and therefore $f/d \notin \mathcal{D}$.

4. Let $C = \Re \setminus D$, where $D \neq \emptyset$ and card $D < c$. Then for every function $f \in \mathcal{D}^{**}$ there exists a function $d \in \mathcal{D}^{**}$ such that $\max(f, d) \in \mathcal{D}^{**}(\Re, C)$. Hence $\max(f, d) \notin \mathcal{D}$.

Lemma 2 *For $g : \Re \longrightarrow \Re$ the following are equivalent.*

1. $\max(f, g) \in \mathcal{D}^{**}$ *for each* $f \in \mathcal{D}^{**}$,

2. for every interval I and for every $y \in \Re$ there exist: a subinterval J of I and a subset A of J with card $A < c$ *and $g(x) \leq y$ for each $x \in J \setminus A$.*

Proof. $2 \Rightarrow 1$ Assume that for $g : \Re \longrightarrow \Re$ the condition 2 holds. Let $F \in \mathcal{D}^{**}$, I be an interval and $y \in \Re$. The there exists subinterval J of I such that $g(x) \leq y$ for each $x \in J \setminus A$, where card $A < c$. Therefore card $(\{x \in J : \max(f(x), g(x)) = y\}) = c$ and $\max(f, g) \in \mathcal{D}^{**}$. $1 \Rightarrow 2$ First notice that if $\max(f, g) \in \mathcal{D}^{**}$ for some $f \in \mathcal{D}^{**}$, then g has the following property:

card $(\{x \in I : g(x) \leq y\}) = c$ for every interval I and each $y \in \Re$.

Indeed, in the other case we have card $\{x \in I : \max(f(x), g(x)) = y\}) < c$ and $\max(f, g) \notin \mathcal{D}^{**}$.

Next suppose that for $g : \Re \longrightarrow \Re$ the condition 2 does not hold. Then there exist a $y_0 \in \Re$ and interval I such that card $(\{x \in J : g(x) > y_0\}) = c$ for every subinterval J of I. Let us put $B = \Re$ and $C = (y_0, \infty)$. Then the condition 1 of Theorem holds. Let $c > y_0$ and let J be a subinterval of I. Since card $(\{x \in J : g(x) \leq c\}) = c$ and $\{x \in J : \max(g(x), c) = c\} \subset \{x \in J : \max(g(x), y) = c$ for some $y \in \Re$, the condition 2 of Theorem holds too. Now we shall verify the condition 3. Let J be a subinterval of I and let $y \in \Re$. If $y > y_0$ then card $(\{x \in J : \max(g(x), y) \in C\}) = c$ (because $\{x \in J : g(x) \leq y_0\} \subset \{x \in J : \max(g(x), y) \in C\}$). If $y \leq y_0$ then $\{x \in J : g(x) > y_0\} \subset \{x \in J : \max(g(x), y) \in C\}$ and therefore card $(\{x \in J : \max(g(x), y) \in C\}) = c$.
Since the conditions 1, 2, 3 of Theorem hold, it follows that $\max(f, g|I) \in \mathcal{D}^{**}(I, C)$ for some $f \in \mathcal{D}^{**}(I, \Re)$. Therefore $\max(f, g) \notin \mathcal{D}^{**}$ for some $f \in \mathcal{D}^{**}$ and 1 does not hold.

Corollary 3 *For every function $g : \Re \longrightarrow \Re$ there exists a function $d \in \mathcal{D}^{**}$ such that* $\max(g, d) \notin \mathcal{D}^{**}$.

Proof. Suppose that the condition 2 of Lemma holds for some $g : \Re \longrightarrow \Re$. Then there exist a closed interval $I_0 \subset Re$ and a subset A_0 of I_0 such that card $A_0 < c$ and $g(x) < 0$ for each $x \in I \setminus A$. We can choose two disjoint, closed subintervals $I(1,1), I(1,2)$ of I_0 and subsets $A(1,1) \subset I(1,1)$ and $A(1,2) \subset I(1,2)$ such that $g(x) < -1$ for each $x \in I(1,i) \setminus A(1,i)$, $i = 1,2$. Assume that for fixed $n \in \mathcal{N}$ we have already chosen a sequence of pairwise disjoint, closed intervals $I(n,j)$ and sets $A(n,j) \subset I(n,j)$, $j = 1,2,\ldots,2^n$ such that card $A(n,j) < c$ and $g(x) < -n$ for each $x \in I(n,j) \setminus A(n,j), j = 1,2,\ldots,2^n$. Now, for every $1 \leq j \leq 2^n$ we choose two disjoint, closed subintervals $I(n+1, 2j-1), I(n+1,2j)$ of $I(n,j)$ and subsets $A(n+1,2j-1), A(n+1,2j)$ such that card $A(n+1,i) < c$ for $i = 2j-1, 2j$ and $g(x) < -(n+1)$ for each $x \in I(n+1,i) \setminus A(n+1,i)$, $i = 2j-1, 2j$. Put $C = \bigcap_{n \in \mathcal{N}} \bigcup_{j=1}^{2^n} I(n,j) \setminus \left(\bigcap_{n \in \mathcal{N}} \bigcup_{j=1}^{2^n} A(n,j) \cup A_0 \right)$. Then $C \neq \emptyset$ and $g(x) < -n$ for each $x \in C$ and $n \in \mathcal{N}$, what is impossible.

References

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WYŻSZA SZKOŁA PEDAGOGICZNA
INSTYTUT MATEMATYKI
Chodkiewicza 30
85-064 Bydgoszcz, Poland

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