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Another use of LR and QR decompositions

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The aim of this paper is to propose two methods, called AL and QL in the sequel, of solving of the eigenvalue problem of a given matrix A. The known LR and QR methods (see e.g. [1]) are not selfcorrecting in the following sense. Each of them constructs a sequence of matrices $A_1 = A, A_2, A_3, \ldots$ where A_{k+1} is defined by means of the decomposition of A_k into the product of a lower and an upper triangular matrices L_k, R_k:

$$
\begin{aligned}
A_k &= L_k R_k, \\
A_{k+1} &= R_k L_k
\end{aligned}
\tag{1}
$$

for LR method and similarly

$$
\begin{aligned}
A_k &= Q_k R_k, \\
A_{k+1} &= R_k Q_k
\end{aligned}
\tag{2}
$$

for QR method, with Q_k being a unitary matrix. In both processes the matrix A_{k+1} depends in fact on A_k only and not on A itself. Thus errors produced during the computation of A_k cannot be corrected in the sucessive steps. The methods we propose do not have such a defect.

Definition 1 *AL method.*
Define $L_0 = I$ (identity matrix). For $k = 0, 1, 2, \ldots$ let L_{k+1}, R_{k+1} be given by equalities

$$AL_k = L_{k+1} R_{k+1}, \tag{3}$$

where L_{k+1} and R_{k+1} are lower and upper matrices respectively, L_{k+1} having $1's$ on its diagonal.

Definition 2 *AQ method.*
Define

$$\begin{aligned} Q_0 &= I \\ AQ_{k+1} &= Q_{k+1}R_{k+1} \ (k = 0, 1, 2, \ldots) \end{aligned}$$

where Q_{k+1}, is a unitary matrix and R_{k+1} is an upper triagular matrix.

Observe that if the sequences L_k, R_k (Q_k,R_k, respectively) converge and $L = \lim L_k$, $R = \lim R_k$, $Q = \lim Q_k$ then

$$AL = LR \quad (AQ = QR, \text{ respectively})$$

i.e. the limit matrix R being similar to A, has the same eigenvalues as A has.

The applicability conditions are the same for both LR and AL methods (for QR and AQ, respectively) for any matrix A.

Theorem 3 *The AL is applicable to a matrix A iff the LR is, i.e. for all $k = 1, 2, 3, \ldots$ there exist matrices $\overline{L}_k$, $\overline{R}_k$ such that*

$$\overline{L}_0 = I, \ \overline{A}L_k = \overline{L}_{k+1}\overline{R}_{k+1} \tag{4}$$

iff there exist matrices L_k, R_k such that

$$A = L_1R_1, \ L_{k+1}R_{k+1} = R_kL_k \tag{5}$$

Moreover, in this case the following equalities hold:

$$\overline{L}_k = L_1L_2 \ldots L_k, \overline{R}_k = R_k \ (k = 1, 2, 3, \ldots) \tag{6}$$

$$L_k = L_{k-1}^{-1}L_k \ (k = 1, 2, 3, \ldots) \tag{7}$$

Proof. Let us assume that LR is applicable to a given matrix A. Then there exist matrices L_k, R_k satisfying (5). An easy induction on k shows that the matrices L_k defined by (6) satisfy the equality (4). Similarly one checks that converse implication holds. So the theorem follows.

Corollary 4 *If the LR method is convergent, then the AL method provides the convergent sequence R_k and thus provides the eigenvalues of A. Conversely, if the AL method is convergent, then the LR is convergent.*

Remark 5 *It is easy to check that for the matrix* $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *the LR method is convergent while AL is not because* $L_k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.

We omit here analogous theorem and corollary dealing with the QR and AQ methods.

Numerical example 6

The result of applying of the AQ method o the matrix $A = (a_{ij})$ with $a_{ij} = 1/(i+j)$ $(i,j = 1,2,3,4)$ is presented in Table 1. The first row is the result of six steps of QR i.e. the diagonal of R_6. The successive steps do not change the result. The second row gives the result of AQ i.e. the diagonal of R_6. It slightly changes its values in the successive steps. The third row gives the exact (rounded to seven decimal digits) values obtained by a longer double precision calculation. As it may be seen about one decimal digit more is obtained by AQ and it looks typical result for an ill-conditioned matrix as A is. The important thing in this example is that the QR method is not able to improve its result in the following steps while the method AQ is.

Table 1

QR:				
1.75191967E+00	3.42929548E-01	3.57418163E-02	2.53089077E-03	1.28749614E-04
4.72968925E-06	1.22896782E-07	2.147377863E-09	2.26187110E-11	1.29858427E-13
AQ:				
1.75191967E+00	3.42929548E-01	3.57418163E-02	2.53089077E-03	1.28749614E-04
4.72968929E-06	1.22896764E-07	2.14747605E-09	2.26804441E-11	1.01232353E-13

QR 1.0885106630E-09	AQ -2.7212766573E-11
QR -1.4089724845E-16	AQ -1.4098724845E-16
QR 1.4861283420E-23	AQ 6.6875775382E-24
QR -3.1183335447E-32	AQ-1.3067302474E-31
QR 8.1300989227E-38	AQ 2.1389153468E-38
QR 0.0000000000E+00	AQ 0.0000000000E+00
QR 0.0000000000E+00	AQ 0.0000000000E+00
QR 0.0000000000E+00	AQ 0.0000000000E+00
QR 0.0000000000E+00	AQ 0.0000000000E+00
QR 0.0000000000E+00	AQ 0.0000000000E+00

$$\det(A) = 0.0000000000E{+}00$$

References.

[1] J. H. Wilkinson, *The algebraic eigenvalue problem*, Oxford, 1965

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