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On the order of a group of automorphisms of a compact bordered Klein surface

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We will prove the announced results by means of NEC groups. An *NEC-group* is a discrete subgroup Γ of the group of isometries $\mathcal{G}$ of the hyperbolic plane $\mathcal{C}^+$ (including those which reverse orientation—reflections and glide reflections) with compact quotient space $\mathcal{C}^+/\Gamma$. Let $\mathcal{G}^+$ denote the subgroup of index 2 in $\mathcal{G}$ consisting of orientation preserving isometries. An NEC group Γ contained in $\mathcal{G}^+$ is called a *Fuchsian group*, and a *proper NEC-group* in the other case. In what follows $\Gamma^+ = \Gamma \cap \mathcal{G}^+$ is the canonical Fuchsian subgroup of an NEC group Γ.

Macbeath [7] and Wilkie [13] associated to every NEC group a signature that has the form

$$(g; \pm; [m_1, \ldots, m_r], \{(n_{i1}, \ldots, n_{i,s_i})_{i=1,\ldots,k}\}) \tag{1}$$

and determines the algebraic structure of the group. The numbers m_i are called *proper periods*, the brackets $(n_{i1}, \ldots, n_{is_i})$ *period cycles* and $g \geq 0$ is called *orbit genus*. The group with signature 1 has the presentation with the following generators

$$\begin{array}{ll} x_i,\ i = 1, \ldots, r, & \\ c_{ij},\ i = 1, \ldots, k,\ j = 0, \ldots, s_i, & \\ e_i,\ i = 1, \ldots, k, & \\ a_i,\ b_i,\ i = 1, \ldots, g & \text{(if the sign is } +\text{)} \\ d_i,\ i = 1, \ldots, g & \text{(if the sign is } -\text{)} \end{array} \tag{2}$$

subject to the relations

1. $x_i^{m_i} = 1$, $i = 1, \dots, r$,

2. $c_{is_i} = e_i^{-1} c_{i0} e_i$, $i = 1, \dots, k$,

3. $c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1} c_{ij})^{n_{ij}} = 1$, $i = 1, \dots, k$; $j = 1, \dots, s_i$,

4. $x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$ if the sign is $+$,
$x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1$ if the sign is $-$.

In what follows these generators are said to be the canonical generators of Γ. It is known that the only elements of finite order in Γ are those that are conjugate to powers of c_{ij}, $c_{ij} c_{ij-1}$, x_i. Every NEC group has a fundamental region associated, whose area depends only on the group. It is given by

$$\mu(\Gamma) = 2\pi \left(\alpha g + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \left(1 - \frac{1}{n_{ij}}\right)\right), \tag{3}$$

where $\alpha = 1$ if the sign is $-$ and $\alpha = 2$ in the other case.

It is known that the necessary and sufficient condition for a group Γ with presentation 2 to be realized as an NEC group with signature 1 is that the right hand side of 3 is greater than 0.

If Γ is a subgroup of finite index in an NEC group Λ, then it is an NEC group itself and the following Hurwitz-Riemann formula holds

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda). \tag{4}$$

An NEC group with signature

$$\left(g; \pm; [-]; \{(-), .\overset{k}{.}., (-)\}\right) \tag{5}$$

($k \geq 1$) is said to be a *bordered surface group* of genus g with k boundary components orientable or non-orientable according as the sign is $+$ or $-$. The number $p = \alpha g + k - 1$ is called the *algebraic genus* of Γ and it equals the algebraic genus of the corresponding Klein surface $X = \mathcal{C}^+/\Gamma$.

It is known [11] that a compact bordered Klein surface of algebraic genus $p \geq 2$ can be represented as $\mathcal{C}^+/\Gamma$, where Γ is a bordered surface group of algebraic genus p. Moreover given a surface so represented, a finite group G is a group of its automorphisms if and only if there exists a proper NEC group Λ containing Γ as a normal subgroup such that $G \cong \Lambda/\Gamma$ [8].

Lemma 1 *The only proper NEC groups with area smaller than* $\pi/6$ *are those which have a signature* $(0; +; [-]; \{(n_1, n_2, n_3)\})$, *where* $5/6 < 1/n_1 + 1/n_2 + 1/n_3 < 1$ *or* $(0; +; [m]; \{(n)\})$, *where* $5/6 < 2/m + 1/n < 1$.

Proof. Straightforward verification.

Lemma 2 *None of the groups listed in the previous lemma admit a bordered surface group as a normal subgroup of finite index.*

Proof. Notice that a canonical Fuchsian subgroup Γ^+ of a bordered surface group Γ is torsion free.

It is easy to check that an NEC group Λ with a signature

$$(0; +; [-]; \{(n_1, n_2, n_3)\})$$

is generated by three reflections c_0, c_1 and c_2 obeying the relations

$$(c_0 c_1)^{n_1} = (c_1 c_2)^{n_2} = (c_0 c_2)^{n_3} = 1.$$

Assume that a group Λ contains a bordered surface group Γ as a normal subgroup. Then a reflection c of Λ belongs to Γ. Reflection c is conjugate to one of the canonical ones, say to c_0 and since Γ is normal in Λ c_0 itself belongs to Γ. Now since $\mu(\Lambda) > 0$, n_1 or n_3 is greater than 2. But then $(c_0 c_1)^2$ or $(c_0 c_2)^2$ is a nontrivial element of finite order in Γ^+ which is torsion-free as we already mentioned, a contradiction.

Now assume that Λ has a signature of the second type. Then Λ is generated by c_0, c_1, and e subject to the relations

$$\begin{aligned} c_0^2 = c_1^2 &= (c_0 c_1)^n = 1, \\ e^m &= 1, \\ e c_0 e^{-1} &= c_1. \end{aligned}$$

As in the previous case we argue that one of c_i belongs to Γ. But then the other one does. So $c_0 c_1$, being an element of order n belongs to Γ^+, a contradiction.

Corollary *The order of a group of automorphisms of a bordered compact Klein surface of algebraic genus* $p \geq 2$ *is bounded above by* $12(p-1)$.

Proof. If a finite group G is a group of automorphisms of a bordered Klein surface of algebraic genus $p \geq 2$ then $G = \Lambda/\Gamma$, where Γ is a bordered surface group of area $2\pi(p-1)$ and by lemma 1 $\mu(\Lambda) \geq \pi/6$. Thus

$$|G| = \mu(\Gamma)/\mu(\Lambda) \leq \frac{2\pi(p-1)}{\pi/6} = 12(p-1).$$

Remark 1 *It turns out that an NEC-group* Γ *with signature*

$$(0, +; [-], \{(3, 2, 2, 2)\})$$

and area $\pi/6$ *is the group which admits bordered surface groups as normal subgroups of a finite index [3], [9] and it was shown in many papers that the bound* $12(p-1)$ *is attained for infinitely many groups [3], [4], [5], [6], [9], [10], [12].*

Remark 2 *Recently it was shown in [2] that the necessary and sufficient condition for an NEC group* Λ *to admit a bordered surface group* Γ *as a normal subgroup of finite index is that* Λ *has a signature with an empty period cycle or with a period cycle with two consecutive periods equal to 2. An NEC group* Λ *with an empty period cycle has clearly area* $\geq \pi/3$ *while it is easy to observe that a period cycle with two consecutive periods equal to 2 in an NEC group with area* $< \pi/3$ *has length four and then* $\mu(\Lambda) \geq \pi/6$. *This gives one more proof of the result in question.*

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