Problemy Matematyczne
14 (1995), 21 - 25

On the sup-measurability of multifunctions whose values are allowed to be noncompact

Grażyna Kwiecińska Włodzimierz Ślęzak

The content of the work was announced during the Summer School on Real Functions Theory 1992. It concerns some generalizations of Zygmunt's theorem on the sup-measurability of multifunctions with the Caratheodory condition. The theorems are given without proofs, which can be found in [5].

Let $(X, \mathcal{M}(X))$ be a measurable space, (Y, d) a complete separable metric space and (Z, ϱ) a metric space. Suppose that $\mathcal{M}(X)$ has the following „projection property":

If $A \in \mathcal{M}(X) \otimes \mathcal{B}_0(Y)$ then $\pi_X(A) = \{x : \exists y \in Y(x,y) \in A\} \in \widehat{\mathcal{M}}(X)$ where $\mathcal{B}_0(Y)$ denotes the σ-field of Borel sets in Y and $\widehat{\mathcal{M}}(X)$ denotes a completion of $\mathcal{M}(X)$.

A multifunction $F : X \times Y \to Z$ is $\mathcal{M}(X)$ sup-measurable if for each $\mathcal{M}(X)$measurable multifunction $G : X \to Y$ with closed values the superposition $\Phi : X \to Z$ defined by formula $\Phi(x) = F(x, G(x)) = \cup_{y \in G(x)} F(x,y)$ is $\mathcal{M}(X)$measurable, i.e. $\Phi^+(B) = \{x \in X : \Phi(x) \subset B\} \in \mathcal{M}(X)$ for each open set $B \subset Z$.

Theorem 1 Let (Z, ϱ) be σcompact. Suppose that $F : X \times Y \to Z$ is a multifunction with $\mathcal{M}(X)$measurable all y-sections and mightly lower semicontinous all x-sections. Then F is $\mathcal{M}(X)$sup-measurable.

The mightly lower semicontinuity of multifunction: $H : Y \to Z$ means that it is lower semicontinuous and for each point $y \in Y$ there exists an open set $V(y) \subset Y$ such that $y \in Cl(V(y))$ and $H\mid_{\{y\}\cup V(y)}$ is upper semicontinuous at y.

Let we observe that the lower semicontinuity of all y-sections of F is not sufficient to the $\mathcal{M}(X)$sup-measurability of F. Consider for example a multifunction $F : R \times R \to R$ defined by formula

$$F(x,y) = \begin{cases} \{0,1\} & \text{if } x \neq y \\ \{1\} & \text{if } x = y \text{ and } x \in A \\ \{0\} & \text{if } x = y \text{ and } x \notin A \end{cases}$$

where $A \subset R$ and $A \notin \mathcal{L}(R)$, i.e. A is a nonmeasurable (in the Lebesgue sense) set. All x-sections of F are lower semicontinuos but not mightly lower semicontinuous and al y-sections are $\mathcal{L}(R)$measurable. If $G(x) = \{x\}$ we have $\Phi(x) = F(x, \{x\}) = \{\chi_A(x)\}$. Φ is in this case the characteristic function of the nonmeasurable set.

Theorem 2 Let Z, ϱ be σ-compact. Let $\mathcal{T}(Y)$ be a topology in Y finer than the metric one such that $(Y, \mathcal{T}(Y))$ is separable. Let us fix some countable dense subset of Y and denote it by S. Suppose that for every point $v \in Y$ there exist a subset $U(v) \in \mathcal{T}(Y)$ such that for each $y \in S$ $B_y = \{v \in Y : y \in U(v)\} \in \mathcal{B}_0(Y, d)$ and for every $vinY$ the family $\mathcal{N}(Y) = \{U(v) \cap K(v, 2^{-n}) : n \in N\}$ forms a filter-base of $\mathcal{T}(Y)$-neighbourghoods of point v. Assume that $F : X \times Y \to Z$ is a closed-valued multifunction whose all y-sections are $\mathcal{M}$measurable and all x-sections are $\mathcal{T}(Y)$-continuous. Then F is $\mathcal{M}(X)$sup-measurable.

Example Let $(Y, d, \leq)$ be a partially ordered metric space such that $(Y, \leq)$ is a partially ordered set and there is a countable dense set S in (Y, d) such that for any $y \in Y$ we have: $y = \lim_{n\to\infty} y_n$, for some sequence $y_n \in S$ and $y_n \geq y$ for $n \in N$. Let $calT(Y)$ be a topology on Y generated by all open sets in (Y, d) and also by all the intervals $I_a = \{y \in Y : y \leq a\}$ for $a \in Y$. This topology fulfils the assumptions of the theorem 2 as it have shown by Dravecky and Neubrunn (see theorem 6.5 on p. 156 in [2]). Fromtheorem 2 we obtain a corollary.

Corollary A multifunction $F : X \times R \to R$ having all x-sections right-continuous (left-continuous) and all y-sections $\mathcal{M}(X)$measurable is $\mathcal{M}(X)$sup--measurable, provided that values of F are closed.

Remark Let $A \subset R$ and $A \notin \mathcal{L}(R)$. Let $F : R \times R \to R$ be given by formula

$$F(x,y) = \begin{cases} \langle 1,2\rangle & \text{if } x \in A \text{ and } y \leq x \\ \langle 1,2) & \text{if } x \notin A \text{ and } y < x \\ \{0\} & \text{in other cases.} \end{cases}$$

Some of its x-sections are right-continuous, remaining are left-continuous, all y-sections are $\mathcal{L}(R)$measurable but it is not $\mathcal{L}(R)$sup-measurable.

This example shows that the assumption of one-side continuity from the same side for all x-sections is essential.

Theorem 3 Let $(X, \mathcal{T}(X))$ be a perfectly normal topological space and let (Z, ϱ) be σ-compact. If $F : X \times Y \to Z$ is a multifunction with $\mathcal{T}(X)$-lower semicontinuous y-section and upper semicontinuous x-sections, then F is $\mathcal{B}_0(X)$sup-measurable.

Remark 2 Let $\mathcal{T}(X)$ be the density topology in X. A multifunction $H : X \to Z$ is called approximately lower semicontinuous if $H^{-}(U) = \{x \in X : H(x) \cap U \neq \emptyset\} \in \mathcal{T}(X)$ for each open set $U \subset Z$. In this case theorem 3 is a generalization Grande's theorem onto the case of multifunctions (see [4], theorem 30).

Let $(Y, d, \mathcal{M}(Y), \mu)$ be a complete separable metric space with σ-finite regular complete measure defined on a σ-field $\mathcal{M}(Y)$ of subset of Y containing $\mathcal{B}_0(Y)$. Let $\mathcal{D}(Y)$ be a differentition basis for the space $(Y, d, \mathcal{M}(Y), \mu)$ (see [1], p. 30) fulfilling the following conditions:

1. $\mathcal{D}(Y) \subset \mathcal{M}(Y)$ is a countable family of sets with nonempty interiors and positive finite measure μ

2. For every point $y \in Y$ there exists a sequence of sets $(I_n)_{n=1}^{\infty}$ from $\mathcal{D}(Y)$ converging to y, i.e. $y \in Int(I_n)$ for $n \in N$ and sequence of diameters of I_n converges to zero when n approaches innfinity.

(*). A set $A \subset Y$ has the property (Z) with respect to $\mathcal{D}(Y)$ if for any point $y \in Y$ there exist an open set $U \subset A$ and a number $\delta > 0$ such that

$$\frac{\mu(J \cap U)}{\mu(J)} > \frac{1}{2}$$

for each set $J \in \mathcal{D}(Y)$ containing y and with diameter less than δ.

Denote by $\mathcal{Z}$ the family of subsets of Y with the property (Z). A multifunction $H : Y \to Z$ has the property (Z) if $H^{-}(G) \in \mathcal{Z}$ and $H^{+}(G) \in \mathcal{Z}$ for each open set $G \subset \mathcal{Z}$.

Theorem 4 Let $(X, \mathcal{B}(X))$ be the Baire space, $(Y, d, \mathcal{M}(Y), \mu)$ let be as in (*) and let Z be σ-compact. Suppose that $F : X \times X \to Z$ is a multifunction such that all its x-sections have the property (Z) with respect to $calD(Y)$ and all its y-functions have the Baire property. Then F is $\mathcal{B}(X)$sup-measurable, where $\mathcal{B}(X)$ denotes the σfield of the subsets of X with Baire property.

Theorem 4 is a generalization of theorem E. Grande and Z. Grande (see [3], theorem 1) onto the case of multifunctions.

Each of these above theorems serve as a kind of generalization of Zygmunt's theorem (see [7], theorem 2):

Theorem (Zygmunt) If all x-sections of multifunction $F : X \times Y \to Z$ with compact values are continuous and all its y-sections are $\mathcal{M}(X)$sup-measurable, then F is $\mathcal{M}(X)$sup-measurable.

References

[1] Bruckner A.M., *Differentiation of integrals*, Amer. Math. Monthly 78 (1979), 1 - 54.

[2] Dravecky J., Neubrunn T., *Measurability of functions of two variables*, Matematicky Časopis 23 no 2, (1973), 147 - 157.

[3] Grande E., Grande Z., *Quelques remarques sur la superposition $F(x, f(x))$*, Fund. Math. CXXI (1984), 199 - 211.

[4] Grande Z., *La mesurabilité des fonctions de deux variables et de la superposition $F(x, f(x))$*, Dissertationes mathematicae CLIX, 1 - 49.

[5] Kwiecińska G., Ślęzak W., *Sup-measurability of multifunctions*, to appear.

[6] Spakowski A., *On Superpositionally Measurable Multifunctions*, Acta Univ. Carolinae, vol. 30 no 2 (1989), 149 - 151.

[7] Zygmunt W., *Remarks on superpositionally measurable multifunctions* (in Russian), Mat. Zamietki (= Matn. Notes) vol. 48 no 3 (1990), 70 - 72.

Streszczenie

O Supermierzalności multifunkcji o nie koniecznie zwartych wartościach

W tym komunikacie, prezentowanym na konferencji w Czecho-Słowacji, sformułowane zostały kryteria na cięcie multifunkcji 2 zmiennych gwarantujące jej superpozycyjną mierzalność względem zupełnej sigma-algebry. W odróżnieniu od wcześniejszych prac Caljuka, Spakowskiego i Zygmunta nie wymaga się ciągłości żadnych cięć, lecz zastępuje się ją pewnymi uogólnieniami jednostronnej ciągłości, silną półciągłością z dołu lub warunkiem (Z) Grandego. W prezentowanych wynikach nie jest istotne, aby wartości rozważanych multifunkcji były zwarte. Ponadto przedstawiono 2 kontrprzykłady, wskazujące, że nie jest możliwe dalsze osłabianie założeń w pewnych narzucających się kierunkach. Pełne dowody znajdują się w [5].

GDANSK UNIVERSITY
INSTITUTE OF MATHEMATICS
Gdańsk-Oliwa 80-952
Stwosza 57
Poland

PEDAGOGICAL UNIVERSITY
INSTITUTE OF MATHEMATCS
Bydgoszcz 85-064
Chodkiewicza 30
Poland