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On the function of oscillation

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Throughout this paper we will use the following denotations, facts and definitions (see [1], [2]):
$I\!R$ will denote the set of all real numbers.

Definition 1 Let $\mathcal{B}_0^+$ be a non empty family of non empty sets $E \subset I\!R$ such that

a$_1$ if $E \in \mathcal{B}_0^+$, then for every $t > 0$, $E \cap (0, t) \in \mathcal{B}_0^+$.

a$_2$ $E_1 \cup E_2 \in \mathcal{B}_0^+$ if and only if $E_1 \in \mathcal{B}_0^+$ or $E_2 \in \mathcal{B}_0^+$.

For every set $E \subset I\!R$ and $x \in I\!R$ we shall write

$$E + x = \{y : \bigvee_{a \in E} (y = a + x)\}, \quad -E = \{y : -y \in E\}.$$

Then the family $\mathcal{B}_0^-$ is defined as

$$\mathcal{B}_0^- = \{E : -E \in \mathcal{B}_0^+\}.$$

For every $x \in I\!R$ let

$$\mathcal{B}_x^+ = \{E : (E - x) \in \mathcal{B}_0^+\}, \mathcal{B}_x^- = \{E : -E \in \mathcal{B}_0^+\}.$$

and $\mathcal{B}_x = \mathcal{B}_0^+\} \cup \mathcal{B}_x^-$. Now let $\mathcal{B} = \bigcup_{x \in I\!R} \mathcal{B}_x$.

Definition 2 The number y_0 is called a $\mathcal{B}$-limit number of a function $f : I\!R \longrightarrow I\!R$ at a point $x_0 \in I\!R$ if for every $\varepsilon > 0$ the set

$$\{x : |f(x) - g| < \varepsilon\}$$

belongs to $\mathcal{B}_{x_0}$.

Let $L_{\mathcal{B}}(f, x)$ denotes the set of all $\mathcal{B}$-limit numbers of a function f at a point x.

For every function $f : I\!R \longrightarrow I\!R$ and every point $x_0 \in I\!R$ there exists at least one $\mathcal{B}$-limit number of the function f at the point x_0. For every $x \in I\!R$ the set $L_{\mathcal{B}}(f, x)$ is closed.

Definition 3 We will say that the family $\mathcal{B}$ fulfils the condition M_0, if for every x_0 and a sequence $\{x_n\}$ such that $x_n \searrow x_0$ and for every sequence of sets $\{B_n\}$ such that $B_n \in \mathcal{B}_{x_n}$ the set $E_0 = \bigcup_{n=1}^{\infty} B_n$ belonges to the family $\mathcal{B}^+_{x_0}$.

It is easy to see, that each familly $\mathcal{B}$ fulfiling the condition M_0 fulfils the condition M_1 too.

Let us write for a bounded function $f : I\!R \to I\!R$.

$$m_{\mathcal{B}}(f, x) \min L^*_{\mathcal{B}}(f, x)$$

$$M_{\mathcal{B}}(f, x) \max L^*_{\mathcal{B}}(f, x),$$

where $L^*_{\mathcal{B}}(f, x) = L_{\mathcal{B}}(f, x) \cup \{f(x)\}$. We shall say that the function f is upper $\mathcal{B}$-semicontinuous (lower $\mathcal{B}$-semicontinous) at a point x_0 if

$$M_{\mathcal{B}}(f, x) \leq f(x_0), \quad m_{\mathcal{B}}(f, x) \geq f(x_0).$$

From theorem 14 ([2]) we infer the following characterization:
for an arbitrary bounded function f, $M_{\mathcal{B}}(f)$, $(m_{\mathcal{B}}(f))$ is upper $\mathcal{B}$-semicontinuous (lower $\mathcal{B}$-semicontinous) if and only if the family $\mathcal{B}$ fulfils the condition M_1. Let $\mathcal{P}(x_0) = \{p \in I\!R \times I\!R : p = (x_0, y)\}$. Let $f : I\!R \to [0, 1]$ be an arbitrary function. Let $\mathcal{B}^+_0 = \mathcal{N}^+_i$ be the family of sets for which 0 is a point of right-sided accumulation. It is obvious that the family $\mathcal{N}$ defines ordinary limit numbers.

We shall write

$$L_{\mathcal{N}}(f, x) = L(f, x)$$

$$L^*_{\mathcal{N}}(f, x) = L^*(f, x) \cup \{f(x)\}.$$

The symbol $\omega_f(x)$ will denotes the oscillation of f at a point x, i.e.

$$\omega_f(x) = \max L^*(f,x) - \min L^*(f,x).$$

Let us put $\Omega_f(y) = \{x : \omega_f(x) geq y\}$ for each $y \in [0,1]$.
The following facts are known for an arbitrary function f:

1. The set $\Omega_f(y)$ is closed on $I\!R$ (with natural topology), for each $y \in [0,1]$.

2. If $y_1 < y_2$, then $\Omega_f(y_2) \subset \Omega_f(y_1)$.

3. The set
$$\bigcup_{y\in[0,1]} (\Omega_f(y) \times \{y\})$$
is closed on the plane $I\!R \times I\!R$.

Let now family $\{\Omega(y)\}_{y\in[0,1]}$ of subsets of $I\!R$ fulfils the following conditions:

α_1 The set $\Omega(y)$ is closed on $I\!R$ (with usual topology), for each $y \in [0,1]$.

α_2 If $y_1 < y_2$, then $\Omega(y_2) \subset \Omega(y_1)$.

α_3 The set
$$\bigcup_{y\in[0,1]} (\Omega(y) \times \{y\})$$
is closed on the plane $I\!R \times I\!R$.

α_4 $\Omega(0) = I\!R$.

In this paper there is given the proof of the following
Theorem For an arbitrary family $\{\Omega(y)\}_{0\leq y\leq 1}$ fulfiling conditions (α_1) – –(α_4) there exists a function $f : I\!R \to [0,1]$ such that for each $0 \leq y \leq 1$ we have:

$$(\alpha_0) \quad \Omega(y) = \Omega_f(y).$$

In the proof we shall apply well known Cantor - Bendizson theorem:
every closed set A can be represented as a sum of two sets A_1, A_2, the first of which consists of all points of condensation of A (it is perfect set) and the second one is denumerable; $A_1 \cap A_2 = \emptyset$.

For each $y \in [0, 1]$ let

$$\Omega(y) = A(y) \cup B(y), \quad A(y) \cap B(y) = \emptyset;$$

where $A(y)$ is perfect and B(y) is denumerable.

Notice, that if for some $y' \in (0, 1], x \in A(y')$ then $x \in A(y')$ then $x \in A(y)$ for each $0 \leq y < y'$. If $x \in B(y'), x$ needn't belong to each $B(y)$ for $0 \leq y < y'$. However, if $x \in B(y'')$ for some $y'' < y'$ then, $x \in B(y)$ for each $y'' < y < y'$. Let us define (for each $a \in I\!R$) the set B_a as follows: $B_a = \{y \in [0, 1] : a \in B(y)\}$. Let F be the set of all $a \in I\!R$, which for B_a is nondegenerate interval.

Lemma The set F is denumerable. **Proof** of lemma.

Suppose that F is a nondenumerable set. For each $a \in F$ let $y_a^{(1)}$ and $y_a^{(2)}$ fulfil the inequalities

$$\inf B_a < y_a^{(1)} < y_a^{(2)} < \sup B_a.$$

For an arbitrary $a \in F$ let

$$B'_a = \{y \in [0, 1] : y_a^{(1)} \leq y \leq y_a^{(2)}\}.$$

For $n \geq 2$ $(n \in I\!N)$, by F_n we will demote the set of all $a \in F$, such that diam $B'_a > \frac{1}{n}$. It is easy to see, that

$$F = \bigcup_{n \geq 2} F_n.$$

It follows immediately from our assumption that starting from a certain positive integer all sets F_n are nondenumerable $(F_n \subset F_{n+1})$. Let F_{n_0} be the nondenumerable set. Then at least one of the points $y_k = \frac{k}{2n_0}$ $(k = 0, 1, 2, \ldots, 2n_0)$, belongs to nondenumerable subfamily of the family $\{B'_a\}_{a \in F_{n_0}}$.

Hence there exists a point $a_0 \in B(y_{k_0})$ which is the point of condensation of $\Omega(y_{k_0})$ and accordingly $a_0 \in A(y_{k_0})$. But the sets $A(y_{k_0})$ and $B(y_{k_0})$ are disjoint, so we have a contradiction.

Let us define the function f. Let $U = Q \cap [0, 1]$, where Q is the set of all rational numbers. For an arbitrary set $\Omega(y)$ $(y \in U)$, $A'(y)$ will denote denumerable and dense set on $\Omega(y)$.

Our function f can be now defined as follows.

$$f(x) = \begin{cases} \sup\{y \in U : x \in A'(y)\} & \text{for } x \in \bigcup_{y \in U} A'(y) \\ 0 & \text{otherwise} \end{cases}$$

It is obvious that 0 is an ordinary limit number of f at each point $x \in I\!R$. First we shall prove the inclusion $\Omega_f(y) \subset \Omega(y)$. Let y_0 be a given point from $(0.1]$. Let $x \in \Omega_f(y_0)$ i.e. $\max L^{*}(f, x) \geq y_0$.

(I) if $f(x) \geq y_0 \geq \max L(f, x)$, then from the definition of f and condition (α_3) it follows, that $x \in \Omega(f(x))$. Since $\Omega(f(x)) \subset \Omega(y_0)$ hence $x \in \Omega(y_0)$.

(II) if $y_0 < \max L(f, x)$, then from the definition of f and condition (α_3) we have

$$x \in \Omega(\max L(f, x)).$$

(Notice that in this case we can only have $y_0 < f(x)$). Since $\Omega(\max L(f, x)) \subset \Omega(y_0)$, therefore $x \in \Omega(y_0)$. We shall now prove the inclusion $\Omega(y) \subset \Omega_f(y)$. Let's take on arbitrary point $y_0 \in U \setminus \{0\}$.

(III) Let $x \in A'(y_0)$, then from the definition of f we obtain $x \in \Omega_f(y_0)$.

(IV) Let $x \in A(y_0) \setminus A'(y_0)$.
From the definition of f we obtain:

$$y_0 \leq \max p_y(\mathcal{P}(x) \cap (\bigcup_{y \in [0,1]} (A(y) x \{y\}))) \leq \max L(f, x) \leq \max L^{*}(f, x)$$

Hence $x \in \Omega_f(y_0)$. Let us take now y_0 from $[0, 1] \setminus U$.

(V) Let $x \in A(y_0)$. Here the proof is similar to (IV.)

(VI) In the case $x \in B(y_0)$ we see at once that $x \in \Omega_f(y_0)$.
Remark The function f defined in this paper is of the second class of Baire.

References

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