Problemy Matematyczne
14 (1995), 3 - 14

On the continuous (A, B)–Darboux functions

Dariusz Banaszewski

1. Introduction

Studying the behavior of real functions Z. Grande [1] showed that if a function f maps a set A which is not an interval into a set $I\!R$ of all reals then the set of all continuous functions from A into $I\!R$ which have not the intermediate value property has nonempty interior in the space of all continuous real functions defined on A (with the uniform metric). The main purpose of this paper is to give a classification in the sense of the category the sets of all continuous, (A, B)–Darboux functions in the space of all continuous functions.

Definitions. Let us establish some terminology to be used later. For each points $a, b \in I\!R$, $a \neq b$ by $I_{(a,b)}$ we mean the interval $(\min\{a, b\}, \max\{a, b\})$. Similarly we define the intervals $I_{[a,b]}$, $I_{(a,b]}$. We denote by $\mathrm{cl}\, X$ the closure of X and $\mathrm{int}\, X$ the interior of X. A set $U \subset I\!R$ is called to be an interval in the set $X \subset I\!R$ if there exists an interval $I \subset I\!R$ such that $U = I \cap X$. If A and B are nonempty subsets of $I\!R$ then we said that $f : A \longrightarrow B$ is a (A, B)–Darboux function whenever for any $x_1, x_2 \in A$ such that $x_1 < x_2$ and $f(x_1) \neq f(x_2)$ and for every point $c \in B \cap I_{(f(x_1), f(x_2))}$ there exists $x \in A \cap (x_1, x_2)$ such that $f(x) = c$. Denote by $\mathcal{D}(A, B)$ $(A \neq \emptyset \neq B)$

the family of all (A, B)–Darboux functions and by $\mathcal{C}(A, B)$ the family of all continuous functions $f : A \longrightarrow B$. By the right (left) hand sided cluster set of f at x we mean

$$K^+(f, x) = \{y \in I\!R;\ \text{there is a sequence } x_n \in A,\ x_n \searrow x \text{ and } f(x_n) \rightarrow y\}$$

$$(K^-(f, x) = \{y \in I\!R;\ \text{there is a sequence } x_n \in A,\ x_n \nearrow x \text{ and } f(x_n) \rightarrow y\})$$

Let ρ be defined by the following formula

$$\rho(f, g) = \min\{1, \sup_{x \in A} |f(x) - g(x)|\}.$$

In this chapter of our article we shall explore the subspace of all (A, B)–Darboux functions in the space $\mathcal{C}(A, B)$ with the metric ρ.

Remark 0.1 *If a nonempty set A is not an interval and the set B has at least tree elements then the set $\mathcal{C}(A, B) \setminus \mathcal{D}(A, B)$ has the nonempty interior.*

Proof. There is a point $a \in I\!R \setminus A$ such that $(-\infty, a) \cap A \neq \emptyset$ and $(a, \infty) \cap A \neq \emptyset$. Let $b = (\sup A \cap (-\infty, a))$ and $c = \inf(A \cap (a, \infty))$. Let $y_1 < y_2 < y_3$ be points of the set B. Let $f : A \longrightarrow B$ be continuous function such that

$$f(x) = \begin{cases} f(x) = y_1 & \text{for } \ x \in A \cap (-\infty, b], \\ f(x) = y_3 & \text{for } \ x \in A \cap [c, \infty), \end{cases}$$

Then for every function $g \in \mathcal{C}(A, B)$ with $\rho(f, g) < \delta$, where $\delta = \frac{1}{2}\min\{y_3 - y_2, y_2 - y_1, 1\}$ g cannot be in $\mathcal{D}(A, B)$. This completes the proof.

Theorem 0.1 *Suppose that A and B are nonempty, $\operatorname{cl} A \setminus A$ is not closed in $I\!R$ and B is dense in itself, then the set $\mathcal{C}(A, B) \cap \mathcal{D}(A, B)$ is nowhere dense in $\mathcal{C}(A, B)$.*

Proof. Fix f belonging to $\mathcal{C}(A, B)$ and positive r. Since $\operatorname{cl} A \setminus A$ is not closed, there exists $a \in A$ which is an accumulation point of $\operatorname{cl} A \setminus A$. We can find $b_0, b_1 \in B$ such that:

$$b_1 \in I_{(b_0, f(a))},$$

$$|b_0 - f(a)| < \frac{r}{2},$$

$$|b_0 - b_1| < \frac{1}{2}|b_0 - f(a)|.$$

Obviously it is possible, since B is dense in itself. By the continuity of f there is an open interval I such that $a \in I$ and $|f(u) - f(a)| < r/4$ for each $u \in I \cap A$. Without loss of generality we may assume that a is a left hand sided accumulation point of $\mathrm{cl}\, A \setminus A$. Choose $x, y \in I \cap (-\infty, a) \setminus A$ such that $x < y$ and $(x, y) \cap A$ is nonvoid. We define a function g as follows:

$$g(u) = \begin{cases} f(u) & \text{if } u \in A \cap [(-\infty, x) \cup (a, \infty)] \\ b_0 & \text{if } u \in A \cap (x, y) \\ f(a) & \text{if } u \in A \cap (y, a] \end{cases}.$$

Evidently $g \in C(A, B)$. Remark that

$$\begin{aligned} |f(u) - g(u)| &\leq |f(u) - f(a)| + |f(a) - g(u)| \\ &< \tfrac{r}{4} + |f(a) - b_0| < \tfrac{r}{4} + \tfrac{r}{2} = \tfrac{3r}{4} && \text{for every } u \in A \cap (x, y), \\ |f(u) - g(u)| &< |f(u) - f(a)| < \tfrac{r}{4} && \text{for every } u \in A \cap (y, a), \\ f(u) &= g(u) && \text{otherwise.} \end{aligned}$$

So $\rho(f, g) \leq 3r/4$. Let $h \in C(A, B)$ be such that $\rho(g, h) < \frac{1}{2}|b_0 - b_1|$. This inequality implies that $\rho(f, h) \leq \rho(f, g) + \rho(g, h) \leq 3r/4 + |b_0 - f(a)|/4 < 3r/4 + r/8 < r$.

Now remark that h cannot be (A, B)-Darboux function. Notice, that $b_1 \in I_{(h(a), h(v))}$ for every $v \in (x, y) \cap A$. For the proof of above we can assume that $f(a) < b_1 < b_0$. If oposite inequalities hold, then the proof is simillar. Fix $v \in (x, y) \cap A$. By definition of h follows that $|h(v) - g(v)| < (b_0 - b_1)/2$. Thus we obtain that

$$\begin{aligned} (b_1 - b_0)/2 < \; & h(v) - g(v) && = h(v) - b_0 \\ b_1/2 + b_1/2 < \; & b_1/2 + b_0/2 && < h(v) \end{aligned}$$

and consequently $b_1 < h(v)$. Moreover, from definitions of b_0, b_1 and h follows:

$$\begin{aligned} |g(a) - h(a)| &< (b_0 - b_1)/2 \\ g(a) - h(a) &> (b_1 - b_0)/2 \\ -h(a) &> b_1/2 - b_0/2 - f(a) \\ h(a) &< b_0/2 + f(a) - b_1/2 < b_1/2 - f(a) - b_1/2 = f(a) < b_1. \end{aligned}$$

Consequently $b_1 \in I_{(h(a),h(v))}$ for every $v \in (x,y) \cap A$.

Notice that $h(t) = b_1$ for no $t \in A \cap (y,a)$. Suppose that $h(t) = b_1$ for some $t \in (y,a) \cap A$. Then $|h(t) - g(t)| = |b_1 - f(a)| > 2|b_0 - b_1|$. This contradics our assumption that $\rho(h,g) < |b_0 - |b_1|/2$. The proof is finished.

Theorem 0.2 *If there exist $a, b \in A$ such that $a < b$ and $[a,b] \cap A$ is of the cardinality smaller than continuum and B is nonempty dense in itself then the set $\mathcal{C}(A,B) \cap \mathcal{D}(A,B)$ is nowhere dense in $\mathcal{C}(A,B)$.*

Proof. Let f be a fixed function from the class $\mathcal{C}(A,B)$ and $r > 0$. Because there exist $a, b \in A$, such that $a < b$ and $[a,b] \cap A$ has the cardinality smaller than continuum thus $\mathbb{R} \setminus f([a,b] \cap A)$ is dense in $\mathbb{R}$. We shall consider two cases.

(A) $(a,b) \cap A \neq \emptyset$,

(B) $(a,b) \cap A = \emptyset$.

(A) Let $x_0 \in (a,b) \cap A$ be a fixed point. Now select $y_1, y_2 \in B$ such that:

$$\begin{gathered} y_1 \text{ and } y_2 \text{ are different than } f(a) \text{ and } f(b), \\ \max\{|f(x_0 - y_1|, |f(x_0) - y_2|\} < r/4, \\ y_2 \in I_{(f(b),y_1)}. \end{gathered}$$

Let U_b be an interval in A such that

- b is the right end point of U_b,
- the left end point of U_b belongs to the set $(x_0, b) \setminus A$,
- $|f(x) - f(b)| < |y_2 - f(b)|/2$ for every $x \in U_b \cap A$.

Let U_a be an interval in A such that a is the left end point of U_a and the right end point of U_a is a element of the set $(a, x_0) \setminus A$. Because $[a,b] \cap A$ has the cardinality smaller than continuum we can choose the points $r_1, r_2 \notin f([a,b] \cap A)$ with the following conditions:

$$\begin{gathered} f(x_0), y_1, y_2 \in (r_1, r_2), \\ |r_2 - r_1| < r/2. \end{gathered}$$

Define U as

$$U = f^{-1}([r_1, r_2]) \cap (a, b) \setminus [U_a \cup U_b],$$

and let us put

$$g(x) = \begin{cases} y_1 & \text{for} \quad x \in U \\ f(x) & \text{for} \quad x \in A \setminus U \end{cases}$$

Remark that g is from the class $\mathcal{C}(A, B)$, but g cannot be in $\mathcal{D}(A, B)$. Indeed, $y_2 \in B$ is between $g(x_0) = y_1, g(b) = f(b)$ and moreover $g(u) \neq y_2$ for every $u \in (x_0, b)$. Let

$$\delta = \frac{1}{2} \min\{|y_1 - y_2|, y_2 - r_1, r_2 - y_2, \frac{1}{2}|y_2 - f(b)|\}. \tag{1}$$

We shall show that $\rho(f, h) < r$ for every function $h \in \mathcal{C}(A, B)$ with $\rho(h, g) < \delta$ and

$$\{h \in \mathcal{C}(A, B); \rho(h, g) < \delta\} \cap \mathcal{D}(A, B) = \emptyset. \tag{2}$$

Let $h \in \mathcal{C}(A, B)$ be such that $\rho(h, g) < \delta$ and $x \in A$ be a fixed point. If $x \notin U$ then $f(x) = g(x)$ and consequently $|f(x) - h(x)| < \delta < r$. Assume that $x \in U$. Then we obtain the following chain of inequalieties:

$$\begin{aligned} |f(x) - h(x)| &\leq |f(x) - g(x)| + |g(x) - h(x)| \\ &< r_2 - r_1 + \delta < \frac{r}{2} + \frac{1}{2}|y_1 - y_2| \\ &< \frac{r}{2} + \frac{1}{2}(|y_1 - f(x_0)| + |f(x_0) - y_2|) < \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Next observe that

$$h([x_0, b]) \cap (y_2 - \delta, y_2 + \delta) = \emptyset \tag{3}$$

Indeed, if $x \in U$ then $|g(x) - y_2| \leq |g(x) - h(x)| + |h(x) - y_2|$ and we conclude that

$$\begin{aligned} |h(x) - y_2| &\geq |g(x) - y_2| - |g(x) - h(x)| = \\ &= |y_1 - y_2| - |g(x) - h(x)| > 2\delta - \delta = \delta. \end{aligned}$$

If $x \in [x_0 - b] \setminus U$, then either $x \in U_b$ or $f(x) \notin [r_1, r_2]$.

Assume that $x \in U_b$. By definitions of U_b $|f(x) - y_2| > |f(b) - y_2|/2 \geq 2\delta$ and notice that

$$\begin{aligned} |h(x) - y_2| &\geq |g(x) - y_2| - |g(x) - h(x)| = \\ &= |f(x) - y_2| - |g(x) - h(x)| > 2\delta - \delta = \delta. \end{aligned}$$

Let $x \in (x_0, b)$ be such that $f(x) \notin [r_1, r_2]$. Since $f(x) = g(x)$ and $y_2 \in (r_1, r_2)$, thus $|g(x) - y_2| > \min\{y_2 - r_1, r_2 - y_2\} \geq 2\delta$ and finally

$$|h(x) - y_2| \geq |g(x) - y_2| - |g(x) - h(x)| > 2\delta - \delta = \delta$$

It is easily seen that $y_2 \in I_{[h(x_0),h(b)]}$ and from (3) we conclude that (2) holds. So the proof is complete in the case (A).

(B) Assume that $(a, b) \cap A = \emptyset$. Let $V \subset I\!R$ be a connected component of A such that $b \in V$. Then V is a nondegenerate interval or $V = \{b\}$.

(i) Assume that V is nondegenerate.
By the continuity of f there exists the maximal interval W in $I\!R$ such that $W \subset V$, b is the left end point of W and $f(W) \subset [f(b) - r/4, f(b) + r/4]$. Of course $f(W) \subset B$ is an interval. Choose points $b_1, b_2 \in B$, $x_0 \in \operatorname{cl} W$, $b_1 \in I_{(b_2, f(a))}$ such that either:

(a) $b_1, b_2 \in f(W)$ and $f(x_0) = b_2$, if $f(W)$ is nondegenerate

or

(b) $b_1, b_2 \in (f(b) - r/4, f(b) + r/4)$ and $x_0 = \sup W$, otherwise.

Let us put

$$g(x) = \begin{cases} b_2 & \text{for} \quad x \in [b, x_0] \cap A \\ f(x) & \text{for} \quad x \in [(-\infty, a] \cup (x_0, \infty)] \cap A \end{cases}$$

Then $\rho(g, f) \leq r/2$ and $g \in C(A, B)$. Notice that $h \in \mathcal{D}(A, B)$ for no function which $\rho(h, g) < \min\{|b_1 - f(a)|, |b_2 - b_1|\} < r/2$. It follows immediately from the fact that $b_1 \in I_{(h(a),h(b))}$ and are not points of between a and b. Thus the proof is finished in this case.

(ii) Now we shall consider the case when $\{b\}$ is connected component of A. First assume that b is an isolated point of A. Let $b_1, b_2 \in B \cap (f(b) - r/4, f(b) + r/4)$, $b_1 \in I_{(f(a),b_2)}$. Define

$$g(x) = \begin{cases} f(x) & \text{for} \quad x \in A \setminus \{b\} \\ b_2 & \text{for} \quad x = b \end{cases}$$

Then $g \in \mathcal{C}(A, B)$. Notice that every function $h \in \mathcal{C}(A, B)$ with $\rho(h, g) < \min\{|f(a) - b_1|, |b_1 - b_2|\}$ is equal to b_1 for no points of $[a, b] \cap A$ and $b_1 \in I_{(h(a), h(b))}$. Thus $h \notin \mathcal{D}(A, B)$. Obviously $\rho(h, f) < r$.

Now assume that b is an accumulation point of A and $\{b\}$ is the connected component of A. Then there is an interval W in A such that $|f(x) - f(b)| < r/4$ for each $x \in W$ and b is a left endpoint of W. Choose $a_1, a_2 \in W \setminus A$, $(a_1, a_2) \cap A \neq \emptyset$ and $b_1, b_2, b_3 \in B \cap (f(b) - r/4, f(b) + r/4)$ such that $b_1 \in I_{(b_2, b_3)}$. Put

$$g(x) = \begin{cases} f(x) & \text{for} \quad x \in A \setminus (a, a_2) \\ b_2 & \text{for} \quad x \in (a_1, a_2) \cap A \\ b_3 & \text{for} \quad x \in (a, a_2) \cap A \end{cases}$$

It is easy to see that $g \in \mathcal{C}(A, B)$. Then every function $h \in \mathcal{C}(A, B)$ with $\rho(h, g) < \min\{|b_2 - b_1|, |b_1 - b_3|\}$ is equal to b_1 for no points of $[b, u] \cap A$ and $b_1 \in I_{(h(b), h(u))}$, where $u \in (a_1, a_2) \cap A$. Moreover $\rho(h, f) < r$. Thus $h \notin \mathcal{D}(A, B)$. This completes the proof.

Theorem 0.3 *If $A \neq \emptyset$ is not an interval, B is a nonvoid dense in itself set which contains no interval then $\mathcal{C}(A, B) \cap \mathcal{D}(A, B)$ is nowhere dense in $\mathcal{C}(A, B)$.*

Proof. By theorems 0.1 and 0.2 we can assume that $\operatorname{cl} A \setminus A$ is closed and for all $a, b \in A$ with $a < b$ the set $[a, b] \cap A$ has the cardinality continuum. Fix a function $f \in \mathcal{C}(A, B)$ and $r > 0$. Since the continuous image of connected set is conected we conclude that f must be constant on every connected component of A. Let $U \neq V$ be components of A such that $\sup U \leq \inf V$.

Denote $f(U) = \{u\}$ and $f(V) = \{v\}$. Let $b_1, b_2, b_3 \in B$ be such that $b_2 \in I_{(b_1, b_3)}$, $\max\{|b_1 - u|, |b_2 - u|\} < r/4$ and $|b_3 - v| < r/4$. Choose $y_1, y_2 \notin B$ such that $u, b_1, b_2 \in I_{(y_1, y_2)}$ and $|y_2 - y_1| < r/2$. Let $a_1 \in [\sup U, \inf V] \setminus A$ be such that:

$$\operatorname{osc}_{[a_1, \sup V] \cap A} f \leq r/4$$

and choose $a_2 > \sup U$, $a_2 \neq a_1$, which fulfill conditions:

$$a_2 \notin A$$

$$\mathrm{osc}_{[\inf V, a_2]} f < r/4$$

(a_2 may be equal to ∞ if V is nobounded). Let g be define as:

$$g(x) = \begin{cases} b_1 & \text{for} \quad x \in A \cap (-\infty, a_1) \cap f^{-1}(I_{(y_1,y_2)}) \\ b_3 & \text{for} \quad x \in A \cap (a_1, a_2) \\ f(x) & \textit{otherwise.} \end{cases}$$

Since $f \in C(A, B)$, each set $A \cap (-\infty, a_1) \cap f^{-1}(I_{(y_1,y_2)})$ and $x \in A \cap (a_1, a_2)$ is both open end closed in A we have that g is continuous. Moreover, it is easy to see that $\rho(g, f) < r/2$. Notice that for every $h \in C(A, B)$ with

$$\rho(h, g) < \min\{|b_1 - b_2|, |b_3 - b_2|, |y_1 - b_2|, |y_2 - b_2|\} = \delta$$

$h(x)$ is not equal to b_2 for no points of interval $[x_u, x_v]$ and $b_2 \in I_{(f(x_u), f(x_v))}$, where $x_u \in U$ and $x_v \in V$. Since $b_2 \in I_{(h(x_u), h(x_v))}$, the result is

$$\{h \in C(A, B);\ \rho(h, g) < \delta\} \subset \{h \in C(A, B);\ \rho(h, f) < r\}$$

and

$$\{h \in C(A, B);\ \rho(h, g) < \delta\} \cap \mathcal{D}(A, B) = \emptyset.$$

The proof is finished.

Remark 0.2 *If $A, B \subset \mathbb{R}$ are nonempty and B contains an isolated point then the set $C(A, B) \cap \mathcal{D}(A, B)$ has nonempty interior.*

Proof. Let $b \in B$ be isolated in B, $r = \min\{1, \mathrm{dist}(B) \setminus \{b\}, b)\}$ and $f : A \longrightarrow B$ be constant and equal to b. Then for every function $f : A \longrightarrow B$ with $\rho(g, f) < r$ we have $g = f$.

Theorem 0.4 *If a nonempty set A is such that the set $\mathrm{cl}\, A \setminus A$ is closed and for each points $a, b \in A$, $a < b$, the set $[a, b] \cap A$ has the cardinality continuum and moreover, the set B contains nondegenerate interval then the set $C(A, B) \cap \mathcal{D}(A, B)$ has nonempty interior in $C(A, B)$.*

Proof. If A is closed then A is an interval and $C(A, B) \subset$D (A,B). So we may assume that $\mathrm{cl}\, A \setminus A \neq \emptyset$. Let $(a_1, a_2, \ldots, a_n, \ldots)$, where $a_i \neq a_j$ for $i \neq j$, $i, j = 1, 2, 3, \ldots$ be a sequence (finite or not) of all unilaterally isolated points of the set $\mathrm{cl}\, A \setminus A$. Let $I = [r_1, r_2] \subset B$ be a nondegenerate compact interval. We shall show that A has the following properties:

(1) if $a \in A$ is a point of accumulation of A from the right (left), then there is an open interval U in $I\!R$ such that $U \subset A \cap (a, \infty)$, $(U \subset A \cap (-\infty, a))$ and a is the left (right) endpoint of U.

(2) if $a \in A$ is isolated from the right (left) then $\inf\{x \in A;\ x > a\} \in \operatorname{cl} A \setminus A$ $(\sup\{x \in A;\ x < a\} \in \operatorname{cl} A \setminus A)$ (of course $\inf\{x \in A;\ x > a\}$ $(\sup\{x \in A;\ x > a\})$ is equal to a_i for some $i \in I\!N$).

(3) if $a \in I\!R$ is an accumulation point of $\operatorname{cl} A \setminus A$ from the right (left), then there is the subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$, such that $a_{n_k} \searrow a$ $(a_{n_k} \nearrow a)$ (obviously $a \in \operatorname{cl} A \setminus A$).

(4) if $a \in I\!R$ is an accumulation point of $\operatorname{cl} A \setminus A$ from the right (left), then $\operatorname{int} A \cap (a, a + \delta) \neq \emptyset$ $(\operatorname{int} A \cap (a - \delta, a) \neq \emptyset)$ for every $\delta > 0$.

We will prove (1). Assume that $a \in A$ is a point of accumulation of A from the right and suppose that there is a sequence $(x_n)_{n=1}^{\infty}$, $x_n \notin A$ for $n \in I\!N$ with $x_n \searrow a$. Let U_n be a components of $I\!R \setminus A$ containing x_n. Denote $\operatorname{cl} U_n = [t_n, v_n]$ for $n \in I\!N$. Then $t_n \notin A$ or $v_n \notin A$. Indeed, if $U_n = \{x_n\}$ $(t_n = x_n = v_n)$ then $t_n \notin A$ and $v_n \notin A$. Moreover if U_n is nondegenerate, then since $[t_n, v_n] \cap A$ has the cardinality of continuum for $t_n, v_n \in A$, either $t_n \notin A$ or $v_n \notin A$. Let

$$u_n = \begin{cases} t_n & \text{if} \quad t_n \notin A \\ v_n & \text{if} \quad v_n \notin A \text{ and } t_n \in A \end{cases}$$

for $n \in I\!N$.

It is evident that $u_n \in \operatorname{cl} A \setminus A$ and $u_n \longrightarrow a$. Because $\operatorname{cl} A \setminus A$ is closed thus $a \notin A$. This contradics our assumption.

For the proof of (2) we need notice that, if $s = \inf\{x \in A;\ x > a\} \notin \operatorname{cl} A \setminus A$, then $s \in A$ and $(x, s) \cap A = \emptyset$ $((s, x) \cap A = \emptyset)$, which is impossible.

We next prove that (3) holds. Let $\delta > 0$ and a be an accumulation point of $\operatorname{cl} A \setminus A$ from the right. It is evident that $(\operatorname{cl} A \setminus A) \cap [a, a + \delta]$ is nowhere dense and by assumption closed. Let U be the open, connected component of $(a, a + \delta) \setminus (\operatorname{cl} A \setminus A)$. Thus the endpoint (left or right) u of U belongs to

$(\mathrm{cl}\, A \setminus A) \cap (a, a+\delta)$ and u is unilaterally isolated of $(\mathrm{cl}\, A \setminus A)$. Hence $u = a_i$ for some $i \in \mathbb{N}$ and **(3)** is proved.

To prove **(4)**, by **(1)** it is sufficient to show that for every $\delta > 0$ there exists an acumulation point $x \in A \cap (a, a+\delta)$ of A. But, it is clear from the cardinality of $A \cap (a, a+\delta)$.

We define d to be $(r_2 + r_1)/2$. Let $f : A \longrightarrow B$ be the function with following properties:

(*i*) $f|U$ is continuous,

(*ii*) if a_i is the left (right) endpoint of U for some $i \in \mathbb{N}$, then $K^+(f, a_i) = I$ $(K^-(f, a_i) = I)$,

(*iii*) $f(U) = I$

for each nondegenerate, connected component U of A.
Moreover,

(*iv*) $f(x) = d$ at each unilaterally isolated point x of A.

By **(1)** and definition of f we obtain that f is continuous. Observe that, if a_i $(i = 1, 2, 3, \ldots)$ is not isolated from the left (right) in A, then the cluster set $K^-(f, a_i)$ $(K^+(f, a_i))$ is equal to I.

Let $i \in \mathbb{N}$ be fixed. From *(ii)* the above condition is true for i, whenever a_i is the endpoint of some connected component of A. So we may assume that a_i is the left hand sided point of accumulation of $\mathrm{cl}\, A \setminus A$. From **(3)** and **(4)** follows that for every $\delta > 0$ there exists a nondegenerate conected component U of A such that $U \subset (a_i - \delta, a_i)$. By the above and *(iii)* our properties is proved.

Let $g \in \mathcal{C}(A, B)$ and $\rho(f, g) < (r_2 - r_1)/4$. We shall show that $g \in \mathcal{D}(A, B)$. Fix $a, b \in A$ such that $a < b$ and $g(a) \neq g(b)$.

Let $c \in I_{[g(a), g(b)]}$. If $a_i \in [a, b]$ for no $i \in \mathbb{N}$, then $[a, b] \subset A$ and $g|[a, b]$ has the Darboux property. Consequently, there exists $t \in [a, b] \cap A$ such that

$g(t) = c$.

Now we shall consider the oposite case. From now on we make the assumption $a_i \in (a, b)$ for some $i \in I\!N$. Since $\rho(f, g) < (r_2 - r_1)/4$, it follows that

$$[d - (r_2 - r_1)/4, d + (r_2 - r_1)/4] \subset K^- (f, a_i)$$

or

$$[d - (r_2 - r_1)/4, d + (r_2 - r_1)/4] \subset K^+ (f, a_i).$$

Without restriction of generality we can assume that $[d - (r_2 - r_1)/4, d + (r_2 - r_1)/4] \subset K^- (f, a_i)$. Moreover, assume that $c \in [d - (r_2 - r_1)/4, d + (r_2 - r_1)/4]$. From the above and **(4)**, there are points $u, v \in [a, b]$ such that $g(u) < d - (r_2 - r_1)/4$, $g(v) > d + (r_2 - r_1)/4$ and $[u, v] \subset A$. Since $c \in (g(u), g(v))$, there exists $t \in (u, v)$ such that $f(t) = c$.

If $c \notin [d - (r_2 - r_1)/4, d + (r_2 - r_1)/4]$ then either $g(a)$ or $g(b)$ is not in $[d - (r_2 - r_1)/4, d + (r_2 - r_1)/4]$. Without loss of generality we can assume that $d + (r_2 - r_1)/4 < c \leq g(a)$. Observe that $a \in \operatorname{int} A$. Suppose for a moment that $a \notin \operatorname{int} A$. From **(1)** a is unilaterally isolated of A and from (iv) $g(a) \in (d - (r_2 - r_1)/4, d + (r_2 - r_1)/4)$, which contradicts our assumptions.

Let U be the component of A such that $a \in U$. We will denote by $[u_1, u_2] = \operatorname{cl} U$. Of course $u_2 \in (a, b)$ and either $u_2 \in A$ or $u_2 \notin A$. Assume that $u_2 \notin A$. Obviously, $u_2 = a_j$ for some $j \in I\!N$.
From (ii) $(d - (r_2 - r_1)/4, d + (r_2 - r_1)/4) \subset K^- (g, u_2)$ and consequently there is $x \in (a, u_2)$ such that $f(x) < d$. Since $[a, x] \subset A$, $c \in (f(x), f(a))$ and $g \in C(A, B)$, thus $g|[a, x]$ has the Darboux property. By the above, there is $t \in (a, x) \subset (a, b)$ with $g(t) = c$.

We will consider the last case. Assume that $u_2 \in A$. Then by **(1)** u_2 is an isolated point of A from the right and *(iv)* implies that $f(u_2) = d$, and finally $g(u_2) \in (d - (r_2 - r_1)/4, d + (r_2 - r_1)/4)$. Therefore $c \in (g(u_1), g(a))$ and $g|[a, u_2]$ has the Darboux property, which completes the proof.

References

[1] Grande Z., *On the Darboux property of restricted functions*, preprint.

PEDAGOGICAL UNIVERSITY
INSTITUTE OF MATHEMATICS
Chodkiewicza 30
85-064 Bydgoszcz
Poland