Problemy Matematyczne
12 (1991), 9–15

On Cartesian Products and Diagonals of Quasi-continuous and Cliquish maps

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1. Let two classes of topological spaces $\{X_s\}$, $\{Y_s\}$ and a class of functions $\{f_s\}$, where $f_s : X_s \longrightarrow Y_s$, be given. The transformation associating to each point $x = \{x_s\} \in \prod_s X_s$ the point $\{f_s(x_s)\} \in \prod_s Y_s$ is called the Cartesian product of functions $\{f_s\}$ and is denoted by the symbol $\prod_s f_s$.

For a class of functions $\{f_s\}$, $f_s : X \longrightarrow Y_s$, where X is also a topological space, the transformation associating to each point $x \in X$ the point $\{f_s(x)\} \in \prod_s Y_s$ is called the diagonal of functions $\{f_s\}$ and is denoted by Δf_s. (See e.g. [2], pp. 108–109)

Let Y be a topological space, as well. We say that a function $f : X \longrightarrow Y$ is quasi-continuous at a point $x_0 \in X$ iff for each open neighbourhood U of the point x_0 and for every open neighbourhood V of the point $f(x_0)$, $U \cap \text{int} f^{-1}(V)$ is non-empty (i.e. $x_0 \in \overline{\text{int} f^{-1}(V)}$). A function is quasi-continuous if it is so at each point of its domain (5, see also [1]).

It is well known that if the functions f_s are continuous then their product and diagonal are continuous. In the first part of the article we investigate the problem whether the product or diagonal of quasi-continuous functions are quasi-continuous. Let us remind first, that (evidently) if a function $f : X \longrightarrow Y$ is quasi-continuous at a point $x_0 \in X$ and $g : Y \longrightarrow Z$ is continuous at the point $f(x_0)$, then the function $g \circ f : X \longrightarrow Z$ is quasi-continuous at x_0.

Let $p_t : \prod_s Y_s \longrightarrow Y_t$ be t–th projection on Y_t for each $t \in S$, i.e. $p_t(\{y_s\}) = y_t$. Remind that p_t is always a continuous and open function.

Proposition 1 *The Cartesian product of functions $\{f_s\}$ is quasi-continuous at a point $\{x_s\}$ iff each function $f_s : X_s \longrightarrow Y_s$ is quasi-continuous at the point x_s.*

Proof. Assume first that the function $f = \prod_s f_s$ is quasi-continuous at a point $\{x_s\}_{s\in S}$. Let us fix $t \in S$. Then the transformation $f_1 = p_t \circ f : \prod_s X_s \longrightarrow Y_t$ is quasi-continuous at the point $\{x_s\}$. Let $W \subset X_t$ be an open neighbourhood of the point x_t and let $V \subset Y_t$ be an open neighbourhood of the point $f_t(x_t)$. Then the set $U = \prod_s U_s$, where $U_s = X_s$ for $s \neq t$ and $U_t = W$ is an open neighbourhood of the point $\{x_s\}$, so $\mathrm{int}(U \cap f_1^{-1}(V) \neq \emptyset$. Since $f_1^{-1}(V) = \{\{z_s\} : f_t(z_t) \in V\}$ then

$$\mathrm{int}(W \cap f_t^{-1}(V)) = p_t(\mathrm{int}(U \cap f_1^{-1}(V)) \neq \emptyset.$$

Hence f_t is guasi-continuous at a point x_t. Assume now that for each $s \in S$ the function f_s is quasi-continuous at a point x_s. Let us fix a neighbourhood U of the point $\{x_s\}$ belonging to the standard basis of $\prod_s X_s$ (i.e. the family of all sets of the form $\prod_s W_s$, where every set W_s is open in X_s and $W_s \neq X_s$ only for a finite number of elements of S) and similarly, let V be a neighbourhood of $\{f_s(x_s)\}_{s\in S}$ from the standard basis of $\prod_s Y_s$. Then $U = \prod_s U_s$ and $V = \prod_s V_s$, where for each $s \in S$, U_s is open in X_s, V_s is open in Y_s, $U_s = X_s$ and $V_s = Y_s$ if $s \notin S_0 = \{s_1, s_2, \ldots, s_n\}$, and $x_s \in U_s$, $f_s(x_s) \in V_s$ if $s \in S_0$. Then for each $s \in S_0$ there exists an open, non-empty set $G_s \subset U_s$ such that $f_s(G_s) \subset V_s$. Then the set $W = \prod_s W_s$, where $W_s = X_s$ for $s \notin S_0$ and $W_s = G_s$ for $s \in S_0$, is open and non-empty, $W \subset U$ and $f(W) \subset V$, hence f is quasi-continuous at the point $\{x_s\}$.

Corollary 1 *Cartesian product of quasi-continuous functions is quasi-continuous.*

Proposition 2 *(a) If* $f = \Delta_s f_s : X \longrightarrow \prod_s Y_s$ *is quasi-continuous at a point* x_0*, then for each* $s \in S$ *the function* f_s *is quasi-continuous at that point.*

(b) There exist functions $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ $(i = 1, 2)$ *that are quasi-continuous but* $f_1 \Delta f_2$ *is not quasi-continuous.*

(c) If $f_s : X \longrightarrow Y_s$*,* $s \neq s_0$ *are continuous at the point* $x_0 \in X$ *and* $f_{s_0} : X \longrightarrow Y_{s_0}$ *is quasi-continuous at* x_0 *then* $f = \Delta_s f_s$ *is quasi-continuous at* x_0*.*

Proof. The part (a) is implied from the fact that $f_s = p_s \circ f$ for each $s \in S$.

(b) Suppose that for arbitrary quasi-continuous functions f_1, f_2 the function $f_1 \Delta f_2$ is quasi-continuous. Since the sum $s : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $s(x, y) = x + y$, is a continuous function, we infer that the class of quasi-continuous functions is closed with respect to addition. This is a contradiction with Grande's result saying that each cliquish function is a sum of four quasi-continuous functions [4]. (See also [7] and [8] for another arguments.)

(c) Let us fix $x_0 \in X$ and an open neighbourhood U of the point x_0, moreover, let $V \subset \prod_s Y_s$ be a neighbourhood of $f(x_0)$ from the standard basis at that point. Then $V = \prod_s V_s$, where V_s is open in Y_s and $V_s = Y_s$ if $s \notin S_0 = \{s_1, \ldots, s_n\}$. If $s_0 \notin S_0$ then x_0 is a point of continuity of each of the functions f_s, $s \in S_0$. Then the set $W = U \cap \bigcap_{s \in S_0} \mathrm{int}(f_s^{-1}(V_s)) = U \cap f^{-1}(V)$ is open and non-empty, $W \subset U$ and $f(W) \subset V$. Hence f is quasi-continuous at the point x_0. Let us assume now that x_0 is a point of discontinuity of f_{s_0} and $s_0 \in S_0$. Let $S_1 = S_0 \setminus \{s_0\}$. Then the set $W_1 = U \cap \bigcap_{s \in S_1} \mathrm{int}(f_s^{-1}(V_s))$ is an open neighbourhood of the point x_0. Since f_{s_0} is quasi-continuous at x_0, there exists a non-empty, open set $W \subset W_1 \subset U$ such that $f_{s_0}(W) \subset V_{s_0}$. Then $f(W) \subset V$ and consequently, f is quasi-continuous at x_0. This completes the proof of quasi-continuity of the function f at x_0.

Corollary 2 *If* $f : X \longrightarrow Y$ *is continuous and* $g : X \longrightarrow Z$ *is quasi-continuous, then* $f \Delta g$ *is a quasi-continuous function.*

2. Let Y be a uniform space i.e.Tychonoff space in which the topology is generated by a uniformity $\mathcal{U}$, see [2], pp. 515–518 or [6], pp.

176–180 for the definitions. Let $f : X \longrightarrow Y$ and $x_0 \in X$. We say that f is cliquish at the point x_0 iff for each $V \in \mathcal{U}$ and for every open neighbourhood $\mathcal{U}$ of x_0 there exists a non-empty open set W such that $W \subset U$ and the diameter of the set $f(W)$ is less than V, i.e. $(y_1, y_2) \in V$ for each $y_1, y_2 \in f(W)$. Let us notice that if Y is a metric space with the metric d (for example the real line with the Euclidean topology) then taking for $\mathcal{U}$ the class

$$\{V_\varepsilon : \varepsilon > 0\}, \quad V_\varepsilon = \{(x,y) : d(x,y) < \varepsilon\}$$

we get the well known notion of a cliquish function with values in a metric space (real cliquish function [1] and [3]).

Notice that for a function $f : X \longrightarrow Y$ the following implications hold:
f is continuous at a point $x_0 \Longrightarrow f$ is quasi-continuous at $x_0 \Longrightarrow f$ is cliquish at x_0.

Let the class $\{(Y_s, \mathcal{U}_s)\}_{s \in S}$ of uniform spaces be given. The family $\mathcal{B}$ of neighbourhoods of the diagonal

$$\{(\{y_s\}, \{y_s\}) : \ y_s \in Y_s\} \subset \prod_s Y_s \times \prod_s Y_s$$

consisted of all sets of the form

$$\{(\{y_s\}, \{z_s\}) : \ (y_{s_i}, z_{s_i}) \in V_{s_i} \ \text{ for } \ i = 1, 2, \ldots, k \ \text{ where } \ k \in N,$$

$$s_1, \ldots, s_k \in S \ \text{ and } \ V_{s_i} \in \mathcal{U}_{s_i} \ \text{ for } \ i = 1, \ldots, k\}$$

induces in $\prod_s Y_s$ a uniformity $\mathcal{U}$-the Cartesian product of $\{\mathcal{U}_s\}$. The topology in $\prod_s Y_s$ generated by this structure coincides with the Tychonoff topology, if we consider the product of topologies generated by the uniformities $\mathcal{U}_s$ $(s \in S)$. (See [2], pp. 531–532 or [6], pp. 182–183.)

Proposition 3 *Let $\{X_s\}_{s \in S}$ be a family of topological spaces and let $\{Y_s\}_{s \in S}$ be a family of uniform spaces. The Cartesian product*

$$\prod_s f_s : \prod_s X_s \longrightarrow \prod_s Y_s$$

is a cliquish function at a point $\{x_s\}$ iff the functions $f_s : X_s \longrightarrow Y_s$ are cliquish at x_s for each $s \in S$.

Proof. Assume first that $f = \prod_s f_s$ is cliquish at $\{x_s\}$. Let us fix t from S, an open neighbourhood $U \subset X_t$ of the point x_t and $V \in \mathcal{U}_t$. Then $V_1 = \{(\{y_s\}, \{z_s\}) : (y_t, z_t) \in V\} \in \mathcal{U}$. Since $W = \prod_s W_s \subset \prod_s X_s$, where $W_s = X_s$ for $s \neq t$ and $W_t = U$ is a neighbourhood of the point $\{x_s\}$, there exists an open set $B = \prod_s B_s$ such that B belongs to the standard basis of $\prod_s X_s$, $B \subset W$ and $\delta(f(B)) < V_1$ (i.e. the diameter of $f(B)$ is less than V_1). Then $B_t \subset U$ and $\delta(f_t(B_t)) < V$, so the function f_t is cliquish at the point x_t. Assume now that for each $s \in S$ the function $f_s : X_s \longrightarrow Y_s$ is cliquish at a point x_s. Let $U = \prod_s U_s$ be a basis neighbourhood of the point $\{x_s\}$, where each U_s, is open in X_s and $U_s = X_s$ for $s \notin S_0 = \{s_1, \ldots, s_n\}$, moreover let V be a set from the uniformity $\mathcal{U}$ for the space $\prod_s Y_s$. We can take that V is of the form $\{(\{y_s\}, \{z_s\}) : (y_{s_i}, z_{s_i}) \in V_{s_i}$ for $i = 1, \ldots, n\}$, where $V_{s_i} \in \mathcal{U}_{s_i}$. From the cliquishness of the functions $f_{s_i}, i = 1, \ldots, n$, the existence of non-empty open sets $W_i \subset U_{s_i}$, such that $\delta(f_{s_i}(W_i)) < V_{s_i}$ is implied. Let $W = \prod_s W_s$ be a base set such that $W_s = X_s$ for $s \notin S_0$ and $W_s = W_i$ for $s = s_i$, $i = 1, \ldots, n$. Then $W \subset U$ and $\delta(f(W)) < V$ and, consequently, f is cliquish at $\{x_s\}$.

Proposition 4 *Let X be a topological space and let $\{Y_s\}$, $s \in S$, be a family of uniform spaces.*

(a) If the diagonal of the functions f_s, $\Delta_s f_s : X \longrightarrow \prod_s Y_s$ is cliquish at a point $x_0 \in X$, then each of maps $f_s : X \longrightarrow Y_s$ is cliquish at that point.

(b) If x_0 has the property that for each $s \in S$ there exists a neighbourhood U_s of x_0 on which the function f_s is cliquish, then the diagonal $\Delta_s f_s$ is cliquish at x_0.

Proof. (a) Assume that $f = \Delta_s f_s : X \longrightarrow \prod_s Y_s$ is cliquish at x_0. Let us choose $t \in S$, an open neighbourhood U of the point x_0 and $V \in \mathcal{U}_t$. Then the set $V_1 = \{(\{y_s\}, \{z_s\}) : (y_t, z_t) \in V_t\}$ belongs to the uniformity $\mathcal{U}$ of the product space $\prod_s Y_s$, and hence there exists a non-empty, open set $W \subset U$ such that $\delta(f(W)) < V_1$, what means that $\delta(f_t(W)) < V$.

(b) Assume now that $x_0 \in X$ and for each $s \in S$ there exists an open neighbourhood U_s of the point x_0 on which the function f_s is cliquish.

Let U be a neighbourhood of the point x_0 and let $V \in \mathcal{U}$. We can assume that V is of the form $\{(\{y_s\},\{z_s\}) : (y_s, z_s) \in V_s \text{ for } s \in S_0\}$, where $S_0 = \{s_1, \dots, s_n\}$ is a finite subset of S and $V_s \in \mathcal{U}_s$ for $s \in S_0$. We can choose (inductively) a sequence of non-empty, open sets $W_i \subset U$ $(i = 1, \dots, n)$ such that:

(1) f_{s_i} is cliquish on W_i,

(2) $W_i \subset \bigcap\{U_s : s \in S_0\} \cap U$,

(3) $W_{i+1} \subset W_i$,

(4) $\delta(f_{s_i}(W_i)) \subset V_{s_i}$.

Let $W = \bigcap_{i=1}^n W_i$. Then $\emptyset \neq W \subset U$ and $\delta(f(W)) < V$, hence f is cliquish at the point x_0.

Corollary 3 *If all functions f_s, $s \in S$ are cliquish then the diagonal $\Delta_s f_s$ is so.*

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