Problemy Matematyczne
12 (1991), 17–24

On the Extending of *Baire 1* Functions

Ewa Łazarow

Let $\mathbb{R}$ be the real line, $\mathbb{N}$ the set of all natural numbers and $\mathcal{B}$ the σ–algebra of subsets of $\mathbb{R}$ having the Baire property and I the σ–ideal of sets of the first category on the real line. For $E \subset \mathbb{R}$, let $\text{int}(E)$, $\text{cl}(E)$ denote, respectively, the interior and the closure of E in the natural topology. In [4] there were introduced notions of I–density point and I–dispersion point of a set E having the Baire property. We recall that 0 is an I–density point of a set $A \in \mathcal{B}$ if and only if, for every increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ of real numbers such that $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{t_{n_m}\}_{m \in \mathbb{N}}$ such that $\left\{\chi_{t_{n_m} \cdot A \cap [-1,1]}\right\}_{m \in \mathbb{N}}$ converges to 1 except on a set belonging to I.

Further, x_0 is an I–density point of $A \in \mathcal{B}$ if and only if 0 is an I–density point of the set $A - x_0 = \{x - x_0 : x \in A\}$, and x_0 is an I–dispersion point of A if and only if x_0 is an I–density point of $\mathbb{R} \setminus A$.

Let $\phi(A)$ denote the set of I–density points of A. It turned out (see [4]) that the family $T_I = \{A \subset \mathcal{B} : A \subset \phi(A)\}$ is a topology. It is called the I–density topology. Continuous functions mapping $\mathbb{R}$ with the topology T_I into $\mathbb{R}$ with the natural topology are called I–approximately continuous. The family of these functions will be denoted by C_I.

For any $x \in \mathbb{R}$, we denote by $\mathcal{P}(x)$ the collection of all intervals $[a, b]$ such that $x \in (a, b)$ and of all sets E of the form

$$E = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{x\}$$

where, for every n,

$$a_n < b_n < a_{n+1} < x < d_{n+1} < c_n < d_n \text{ and } x \in \phi(E).$$

In [2] there was introduced a topology τ which consists of all sets $U \in T_I$ such that if $x \in U$, then there exists a set $P \in \mathcal{P}(x)$ such that $P \subset \text{int}(U) \cup \{x\}$. It was proved that τ is the coarsest topology for which all I-approximately continuous functions are continuous.

For any subset $M \subset \mathbb{R}$, define $\Delta(M)$ as the set of all x such that, for each $P \in \mathcal{P}(x)$, we have $0 \neq P \cap M \neq \{x\}$.

We shall need the following theorems and lemmas.

Theorem 1 **(2)** *Let $X \subset \mathbb{R}$. Then τ-$\text{cl}(X) = X \cup \Delta\text{cl}(X) \subset \text{cl}(X)$. Moreover, x_0 is a limit point of X in the τ-topology if and only if $x_0 \in \Delta(\text{cl}(X))$.*

Theorem 2 **(3)** *Let $G \subset \mathbb{R}$ be an open set with respect to the natural topology. Then 0 is an I-density point of G if and only if, for every natural number n, there exist a natural number k and a real number $\delta > 0$ such that, for each $h \in (0, \delta)$ and for each $i \in \{1, \ldots, n\}$, there exist two natural numbers $j_r, j_l \in \{1, \ldots, k\}$ such that*

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j_r - 1}{nk} \right) h, \left(\frac{i-1}{n} + \frac{j_r}{nk} \right) h \right) = \emptyset$$

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j_l}{nk} \right) h, -\left(\frac{i-1}{n} + \frac{j_l - 1}{nk} \right) h \right) = \emptyset.$$

Lemma 1 *Let $A \subset \mathbb{R}$. Then $0 \in \Delta(\text{cl}(A))$ if and only if there exists a natural number n such that, for each $k \in \mathbb{N}$ and for each real number $\delta > 0$, there exist $h \in (0, \delta)$ and $i \in \{1, \ldots, n\}$ such that for each $j \in \{1, \ldots, k\}$,*

$$A \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk} \right) h, \left(\frac{i-1}{n} + \frac{j}{nk} \right) h \right) \neq \emptyset$$

or

$$A \cap \left(- \left(\frac{i-1}{n} + \frac{j}{nk} \right) h,\ - \left(\frac{i-1}{n} + \frac{j-1}{nk} \right) h \right) \neq \emptyset.$$

Proof. Necessity. Suppose that this is not the case. Then, for every natural n, there exist a natural number k and a real number $\delta > 0$ such that, for each $h \in (0, \delta)$ and for each $i \in \{1, \ldots, n\}$ there exist $j_r(i,h), j_l(i,h) \in \{1, \ldots, k\}$ such that

$$A \cap \left(\left(\frac{i-1}{n} + \frac{j_r - 1}{nk} \right) h,\ \left(\frac{i-1}{n} + \frac{j_r}{nk} \right) h \right) = \emptyset$$

and

$$A \cap \left(- \left(\frac{i-1}{n} + \frac{j_l}{nk} \right) h,\ - \left(\frac{i-1}{n} + \frac{j_l - 1}{nk} \right) h \right) = \emptyset.$$

Let $n \in \mathbb{N}$. Now, we shall define the family of sets $\{P_m^{ij}\}$ where $m \in \mathbb{N}$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$. For each natural number $i \in \{1, \ldots, n\}$, we shall say that $h \in P_m^{ij}$ if and only if j is the above-described natural number $j_r(h,i)$, and $m \in \mathbb{N}$ is such that

$$\delta \cdot \left(\frac{(i-1)k + j - 1}{(i-1)k + j} \right)^m \leq h < \delta \cdot \left(\frac{(i-1)k + j - 1}{(i-1)k + j} \right)^{m-1}.$$

We observe that the sets $\{P_m^{ij}\}_{m \in \mathbb{N}}$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, k\}$ have the following properties:

(i) $\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} P_m^{ij} = (0, \delta)$ for all $i \in \{1, \ldots, n\}$;

(ii) if $h_1, h_2 \in P_m^{ij}$, then

$$\left(\frac{(i-1)k + j - 1}{nk} h_1\, ,\ \frac{(i-1)k + j}{nk} h_1 \right) \cap$$

$$\cap \left(\frac{(i-1)k + j - 1}{nk} h_2\, ,\ \frac{(i-1)k + j}{nk} h_2 \right) \neq \emptyset;$$

(iii) if $P_m^{ij} \neq \emptyset$ and $a_m^{ij} = \inf P_m^{ij}$, $b_m^{ij} = \sup P_m^{ij}$, then

$$\bigcup_{h \in p_m^{ij}} \left(\frac{(i-1)k+j-1}{nk} h \,,\ \frac{(i-1)k+j}{nk} h \right) =$$

$$= \left(\frac{(i-1)k+j-1}{nk} a_m^{ij} \,,\ \frac{(i-1)k+j}{nk} b_m^{ij} \right);$$

(iv) $\left(\frac{(i-1)k+j-1}{nk} a_m^{ij} \,,\ \frac{(i-1)k+j}{nk} b_m^{ij} \right) \cap A = \emptyset,$

where a_m^{ij}, b_m^{ij} are described above.

To prove the above statements, see [2], theorem 2. Let

$$r = (i-1)k + j, \quad c_m^{ij} = \frac{r-1}{nk} a_m^{ij} + \frac{1}{3nk} a_m^{ij} \text{ and } d_m^{ij} = \frac{r}{nk} b_m^{ij} - \frac{1}{3nk} b_m^{ij}.$$

Then

$$\left[c_m^{ij}, d_m^{ij}\right] \subset \left(\frac{r-1}{nk} a_m^{ij}, \frac{r}{nk} b_m^{ij} \right)$$

and, for any $m, m' \in \mathbb{N}$, for which $|\, m - m' \,| \neq 1$,

$$\left[c_m^{ij}, d_m^{ij}\right] \cap \left[c_{m'}^{ij}, d_{m'}^{ij}\right] = \emptyset.$$

For any $i \in \{1, \ldots, n\}$, and $j \in \{1, \ldots, k\}$, let $F_{ij} = \bigcup_{m=1}^{\infty} \left[c_m^{ij}, d_m^{ij}\right]$ and

$$P^+ = \bigcup_{m=1}^{\infty} \left(\left[\frac{1}{m+1}, \frac{1}{m} \right] \cap \bigcup_{n=1}^{m} \bigcup_{i=1}^{n} \bigcup_{j=1}^{k} F_{ij} \right) \cup \{0\}.$$

Then P^+ is a perfect set, $P^+ \cap \mathrm{cl}(A) = \{0\}$, and 0 is a right-hand I–density point of P^+ (see [2], theorem 2). In a similar way we can find a perfect set P^- such that $P^- \cap \mathrm{cl}(A) = \{0\}$, and for which 0 is a left-hand I–density point. Let $P = P^+ \cup P^-$. Then P is perfect in the natural topology, $P \cap \mathrm{cl}(A) = \{0\}$ and $0 \in \phi(P)$, which gives a contradiction since, for each $P \in \mathcal{P}(0)$, $\{0\} \neq P \cap \mathrm{cl}(A) \neq \emptyset$.

Sufficiency. We suppose that there exists $P \in \mathcal{P}(0)$ such that $P \cap \mathrm{cl}(A) = \{0\}$. Then $\mathbb{R} \setminus P \supset A \setminus \{0\}$. By assumption, we have that there exists $n \in \mathbb{N}$ such that, for each $k \in \mathbb{N}$ and for each real $\delta > 0$, there exist $h \in (0, \delta)$ and $i \in \{1, \ldots, n\}$ such that, for any $j_r, j_l \in \{1, \ldots, k\}$,

$$\mathbb{R} \setminus P \cap \left(\left(\frac{i-1}{n} + \frac{j_r - 1}{nk}\right) h \, , \, \left(\frac{i-1}{n} + \frac{j_r}{nk}\right) h\right) \supset$$

$$A \cap \left(\left(\frac{i-1}{n} + \frac{j_r - 1}{nk}\right) h \, , \, \left(\frac{i-1}{n} + \frac{j_r}{nk}\right) h\right) \neq \emptyset$$

or

$$\mathbb{R} \setminus P \cap \left(-\left(\frac{i-1}{n} + \frac{j_l}{nk}\right) h \, , \, -\left(\frac{i-1}{n} + \frac{j_l - 1}{nk}\right) h\right) \supset$$

$$A \cap \left(-\left(\frac{i-1}{n} + \frac{j_l}{nk}\right) h \, , \, -\left(\frac{i-1}{n} + \frac{j_l - 1}{nk}\right) h\right) \neq \emptyset.$$

Thus, by lemma 1, 0 is not an I–dispersion point of $\mathbb{R} \setminus P$ which gives a contradiction since $0 \in \phi(P)$. So, the lemma is proved.

We shall use the above lemma for each $x \in \mathbb{R}$ by translating the set if necessary.

Theorem 3 *Let $A \subset [0,1]$. Each Baire one function restricted to A can be extended to $[0,1]$, resulting in an I–approximately continuous function, if and only if $\Delta(\mathrm{cl}(A)) = \emptyset$.*

Proof. Necessity. Let $A \subset [0,1]$ be such that $\Delta(\mathrm{cl}(A)) \neq \emptyset$. We may assume that $0 \in \Delta(\mathrm{cl}(A_1))$ and $A_1 = \{x \in A : x > 0\}$. By lemma 2, we know that there exists a natural number $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ and for each real $\delta > 0$, there exist $h \in (0, \delta)$ and $i \in \{1, \ldots, n\}$, such that, for each $j \in \{1, \ldots, k\}$,

$$A_1 \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) h \, , \, \left(\frac{i-1}{n} + \frac{j}{nk}\right) h\right) \neq \emptyset.$$

We shall define sequences $\{x_p\}_{p \in \mathbb{N}} \subset A_1, \left\{x_p'\right\}_{p \in \mathbb{N}} \subset A_1$ such that for each $p \in \mathbb{N}$, $x_p \neq x_p'$ and

$$0 \in \Delta\left(\mathrm{cl}\left(\{x_p\}_{p \in \mathbb{N}}\right)\right) \cap \Delta\left(\mathrm{cl}\left(\left\{x_p'\right\}_{p \in \mathbb{N}}\right)\right).$$

Let $k=1$. For $\delta=1$, there exist $h_1\in(0,\delta)$ and $i\in\{1,\dots,n\}$ such that $A_1\cap\left(\frac{i-1}{n}h_1\,,\,\frac{i}{n}h_1\right)\neq\emptyset$.

Let $x_1\in A_1\cap\left(\frac{i-1}{n}h_1\,,\,\frac{i}{n}h_1\right)$. Then for $\delta=x_1$, there exist $h_1'\in(0,\delta)$ and $i\in\{1,\dots,n\}$ such that $A_1\cap\left(\frac{i-1}{n}h_1'\,,\,\frac{i}{n}h_1'\right)\neq\emptyset$.

Let $x_1'\in A_1\cap\left(\frac{i-1}{n}h_1\,,\,\frac{i}{n}h_1\right)$. Assume that the sequences

$$\{x_p\}_{p\le\frac{r(r+1)}{2}}\,,\quad\left\{x_p'\right\}_{p\le\frac{r(r+1)}{2}}\,,\quad\{h_p\}_{p\le r}\,,\quad\left\{h_p'\right\}_{p\le r}\,,$$

where $r\in\mathbb{N}$, have been defined.

Let $k=r+1$. For $\delta=x'_{\frac{r(r+1)}{2}+1}$, there exist $h_{r+1}\in(0,\delta)$ and $i\in\{1,\dots,n\}$ such that for each $j\in\{1,\dots,r+1\}$,

$$A_1\cap\left(\left(\frac{i-1}{n}+\frac{j-1}{n(r+1)}\right)h_{r+1},\ \left(\frac{i-1}{n}+\frac{j}{n(r+1)}\right)h_{r+1}\right)\neq\emptyset.$$

Let

$$x_{\frac{r(r+1)}{2}+j}\in A_1\cap\left(\left(\frac{i-1}{n}+\frac{j-1}{n(r+1)}\right)h_{r+1},\ \left(\frac{i-1}{n}+\frac{j}{n(r+1)}\right)h_{r+1}\right)$$

for each $j\in\{1,\dots,r+1\}$.

Then for $\delta=x_{\frac{r(r+1)}{2}+1}$, there exist $h'_{r+1}\in(0,\delta)$ and $i\in\{1,\dots,n\}$ such that for each $j\in\{1,\dots,r+1\}$,

$$A_1\cap\left(\left(\frac{i-1}{n}+\frac{j-1}{n(r+1)}\right)h'_{r+1},\ \left(\frac{i-1}{n}+\frac{j}{n(r+1)}\right)h'_{r+1}\right)\neq\emptyset.$$

Let

$$x'_{\frac{r(r+1)}{2}+j}\in A_1\cap\left(\left(\frac{i-1}{n}+\frac{j-1}{n(r+1)}\right)h'_{r+1},\ \left(\frac{i-1}{n}+\frac{j}{n(r+1)}\right)h'_{r+1}\right)$$

for each $j\in\{1,\dots,r+1\}$.

Now, we shall prove that

$$0\in\Delta\left(\mathrm{cl}\left(\{x_p\}_{p\in\mathbb{N}}\right)\right)\cap\Delta\left(\mathrm{cl}\left(\left\{x_p'\right\}_{p\in\mathbb{N}}\right)\right).$$

Let P be a perfect set such that $P \cap \{x_p\}_{p\in\mathbb{N}} = \emptyset$ and let $G = \mathbb{R} \setminus P$. For any $k \in \mathbb{N}$ and $\delta = x'_{\frac{(k-1)(k-2)}{2}+1}$, there exist $h_k \in (0, \delta)$ and $i \in \{1, \ldots, n\}$ such that for each $j \in \{1, \ldots, k\}$,

$$\{x_p\}_{p\in\mathbb{N}} \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) h_k \,,\, \left(\frac{i-1}{n} + \frac{j}{nk}\right) h_k\right) \neq \emptyset.$$

Let $\{h_{k_s}\}_{s\in\mathbb{N}}$ be a subsequence of $\{h_k\}_{k\in\mathbb{N}}$ corresponding to the same $i \in \{1, \ldots, n\}$. Then for each subsequence $\left\{h_{k_{s_r}}\right\}_{r\in\mathbb{N}}$ of $\{h_{k_s}\}_{s\in\mathbb{N}}$

$$\limsup_{r\to\infty} \frac{1}{h_{k_{s_r}}} \cdot G$$

is residual in $\left[\frac{i-1}{n}, \frac{i}{n}\right] \subset [0,1]$.

Thus 0 is not I–dispersion point of the set G and, thereby, of the set P. Therefore for each $P \in \mathcal{P}(0)$, $P \cap \{x_p\}_{p\in\mathbb{N}} \neq \emptyset$.

In a similar way we can prove that $0 \in \Delta\left(\mathrm{cl}\left(\left\{x'_p\right\}_{p\in\mathbb{N}}\right)\right)$.

Let

$$g(x) = \begin{cases} 1 & \text{at } x = x_n \quad \text{for} \quad n = 1, 2, \ldots \\ 0 & \text{at } x \notin \{x_n\}_{n\in\mathbb{N}} \end{cases}$$

The function g is *Baire 1*, since for each $n \in \mathbb{N}$,

$$x_{\frac{(n+1)n}{n}+1} < \ldots < x_{\frac{(n+1)n}{2}+n} < x'_{\frac{n(n-1)}{2}+1} < \ldots <$$

$$< x'_{\frac{n(n+1)}{2}+(n-1)} < x_{\frac{(n-1)(n-2)}{2}+1}.$$

Let $f : [0,1] \longrightarrow \mathbb{R}$ and $f(x) = g(x)$ for all $x \in A$. We suppose that $f \in C_I$. Then the sets $\{x : f(x) > 0\} \in \tau$, $\{x : f(x) < 1\} \in \tau$ and $0 \in \{x : f(x) > 0\} \cup \{x : f(x) < 1\}$. We suppose that $0 \in \{x : f(x) > 0\}$. Then there exists $P \in \mathcal{P}(0)$ such that $P \subset \mathrm{int}\{x : f(x) > 0\} \cup \{0\}$. This is a contradiction since $\emptyset \neq P \cap \left\{x'_n\right\}_{n\in\mathbb{N}} \subset \{x : f(x) \leq 0\}$. In a similar way we can show that $0 \notin \{x : f(x) < 1\}$. Thus $f \notin C_I$.

Sufficiency. It results from the following theorem [1; 3–e, 21–a, p. 121]: Let τ be a fine topology on a metric space P having the Lusin–Menchoff property. Let M be a τ–isolated G_δ subset of P. If f is a real *Baire 1* function on M, then f can be extended to a real τ–continuous

Baire 1 function on P.

References

[1] Lukes J., Maly J., Zajicek L., *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Mathematics 1189, Springer Verlag.

[2] Lazarow E., *The Coarsest Topology for I–approximately continuous function*, CMUC 27, (4), (1986).

[3] Lazarow E., *On the Baire Class of I–approximately Derivatives*, Proc. Amer. Math. Soc., Vol. 100, (No. 4), (1987).

[4] Poreda W., Wagner-Bojakowska E., Wilczyński W., *A category analogue of the density topology*, Fund. Math. CXXV (1985).

UNIVERSYTET ŁÓDZKI
INSTYTUT MATEMATYKI
Banacha 22
90–238 Łódź, Poland

WYŻSZA SZKOŁA PEDAGOGICZNA
INSTYTUT MATEMATYKI
Chodkiewicza 30
85–064 Bydgoszcz, Poland