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On Darboux Multifunctions

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First let us introduce some of the notions well known in the theory of real functions which can be extended quite easily for real multifunctions i.e. multifunctions defined on the set $\mathbb{R}$ of all real numbers with the values contained in $\mathbb{R}$ too.

By $L(F,x), L^+(F,x), L^-(F,x)$ we shall denote, as usual, the set of all limit points (numbers), right-sided limit points, left-sided limit points of the multifunction F at the point x, i.e. the set of all points y such that there exists a sequence $((x_n, y_n))$ such that

$$y_n \in F(x_n), \quad (x_n, y_n) \longrightarrow (x,y) \text{ and } \; x_n \neq x, \quad x_n > x, \quad x_n < x,$$

respectively.

For $L^*(F,x)$ we shall denote by $\mathcal{L}^*(F,x)$ the set $\{x\} \times L^*(F,x)$, where $*$ stands for the sign $+$, $-$, or for empty sign. To denote that each point y belonging to $F(x)$ is greater than c, we shall write $F(x) > c$.

Repeating the arguments from the article [3] we can notice that

$$\mathrm{ls}_{x>x_0} \mathcal{L}(F,x) \subseteq \mathcal{L}^+(F,x_0).$$

where ls denotes the limit superior of some family of sets in $\mathbb{R}^2$.

Definition 1 A real multifunction F is said to have the Darboux property iff $F(C)$ is connected for every connected subset C of $\mathbb{R}$.

Definition 2 A real multifunction F is said to have the Darboux property at a point $x_0 \in \mathbb{R}$ from the right side iff

(i) $F(x_0)$ is a connected subset of $\mathbb{R}$,

(ii) $F(x_0) \cap L^+(F, x_0) \neq \emptyset$,

(iii) for each $c \in (\inf L^+(F, x_0), \sup L^+(F, x_0))$ and $\delta > 0$ there is $x \in (x_0, x_0 + \delta)$ such that $c \in F(x)$.

In the analogous way we define left-hand sided points of Darboux of a multifunction, and we say that a multifunction has Darboux property at a point if it has Darboux property from both sides at that point.

Lemma Let F be a real multifunction of a real variable with connected values fulfilling the following condition

$$F(x) \cap L^+(F, x) \neq \emptyset, \quad F(x) \cap L^-(F, x) \neq \emptyset.$$

If there exists $c \in \mathbb{R}$ such that

(1) $c \notin F(x)$ for each $x \in \mathbb{R}$,

(2) $F(\mathbb{R}) \cap (c, \infty) \neq \emptyset \neq F(\mathbb{R}) \cap (-\infty, c)$,

then the sets

$$E_1 = \{x \in \mathbb{R} : \quad F(x) < c\}$$
$$E_2 = \{x \in \mathbb{R} : \quad F(x) > c\}$$

are complementary and $K = \operatorname{Fr}(E_1)$ is a perfect set and the sets $K \cap E_i$ are dense in K for $i = 1, 2$.

Proof. Suppose that K is not a perfect set. Since it is closed, then it is not dense in itself. There exists a point $x_0 \in K$ which is isolated in K. There exist two numbers a, b such that $a < b$ and

$$[a, b] \cap K = \{x_0\}.$$

Each $x \in (a, b)$, $x \neq x_0$ belongs to $\operatorname{Int}(E_1)$ or $\operatorname{Int}(E_2)$. Assume that $a \in E_1$. Then the set $C_1 = \{x \in [a, b] : \quad [a, x] \subseteq E_1\}$ is nonempty. Let

$$a_1 = \sup C_1.$$

If $a_1 < x_0$, then $a_1 \in K$, a contradiction, hence $a_1 \geq x_0$. We would get a contradiction, if a_1 were greater than x_0, so $a_1 = x_0$.

In the analogous manner we can prove that

(3) $\quad x_0 = \inf\{x \in [a,b] : \ [x,b] \subseteq E_2\}$ if $F(b) > c$,

or

(4) $\quad x_0 = \inf\{x \in [a,b] : \ [x,b] \subseteq E_1\}$ if $F(b) < c$.

Since $x_0 \in K$, then (4) is impossible, so must (3) hold. We know that $x_0 \in E_1 \cup E_2$. Assume that, for example, $x_0 \in E_1$. Thus $F(x_0) < c$ and

$$F(x_0) \cap L^+(F, x_0) \neq \emptyset.$$

Since in view of (3)

$$L^+(F, x_0) = L(F|_{E_2}, x_0),$$

then we have come to a contradiction. Thus the set K perfect.
Suppose now that the set $K \cap E_1$ is not dense in K. Then there is $x_0 \in K$ and an interval (a, b) such that

(5) $$\{(a,b) \cap K \cap E_1\} \setminus \{x_0\} = \emptyset.$$

Since K is dense in itself then one can choose a and b such that $a \in K$ and $b \in K$ and moreover also $a \in K \cap E_2$, $b \in K \cap E_2$. Then

$$\sup\{x \in [a,b] : \ [a,x] \subseteq E_2\} = b,$$

what contradicts to the fact that $x_0 \in K = \mathrm{Fr}(E_2) = \mathrm{Fr}(E_1)$.

Theorem 1 *A real multifunction has Darboux propety if and only if it has Darboux property at each point of* $\mathbb{R}$.

Proof. Suppose first that a multifunction F has Darboux property but is not Darboux at some point $x_0 \in \mathbb{R}$. Assume that F is not Darboux at x_0 from the right side. Then there exist a positive δ and

$$c \in (\inf L^+(F, x_0), \sup L^+(F, x_0))$$

such that

$$c \notin F(x) \quad \text{for} \quad x \in (x, x_0 + \delta)$$

but there are x_1, x_2 in the interval $(x_0, x_0 + \delta)$ such that

$$F(x_1) < c \quad \text{and} \quad F(x_2) > c.$$

Thus the set $F([x_1, x_2])$ is not connected what is impossible.

Assume now that F has Darboux property at each point of $\mathbb{R}$ and suppose that F has not Darboux property. Then there is a connected set C such that $F(C)$ is not connected. The supposition that C is a singleton is impossible. Then there is a point $c \in \mathbb{R}$ such that $c \notin F(C)$, what means that

$$F(C) = [F(C) \cap (-\infty, c)] \cup [F(C) \cap (c, +\infty)]$$

and the sets $F(C) \cap (-\infty, c)$, $F(C) \cap (c, +\infty)$ are nonempty and separated. There are points $x_1, x_2 \in C$ such that

$$F(x_1) < c < F(x_2) \quad \text{and, for example,} \quad x_1 < x_2.$$

To simplify the denotations, assume that $F_1 = F|_{[x_1,x_2]}$. Of course, $F_1([x_1, x_2])$ is not connected and $F_1 \subseteq A_1 \cup A_2$, where

$$A_1 = [x_1, x_2] \times (-\infty, c),$$

$$A_2 = [x_1, x_2] \times (c, +\infty).$$

The sets A_1 and A_2 are disjoint and open in $[x_1, x_2] \times \mathbb{R}$. Since $F(x_1)$ is a connected set, then for each $x \in [x_1, x_2]$,

$$F(x) \subseteq (c, +\infty) \quad \text{or} \quad F(x) \subseteq (-\infty, c)$$

and

$$F_1 \cap A_1 \neq \emptyset \neq F_1 \cap A_2.$$

Let for $i = 1, 2$,

$$E_i = \{x \in [x_1, x_2] : \ F(x) \subseteq A_i\}.$$

Of course, $E_1 \cup E_2 = [x_1, x_2]$, $E_i \neq \emptyset$, $E_1 \cap E_2 = \emptyset$. Let

$$K = \mathrm{Fr}(E_1)$$

where Fr denotes the boundary of a set with respect to the interval $[x_1, x_2]$. In view of the lemma the set K is perfect and $K \cap E_i$ are dense in K for $i = 1, 2$.

$$F(x) < c \quad \text{for each } x \in K \cap E_1,$$

then

$$L(F,x) \leq c$$

and

$$L(F,x) \geq c \text{ for } x \in K \cap E_2.$$

Let $x_0 \in K$, for example $x_0 \in E_1$. Since $K \cap E_2$ is dense in K, then there exists a sequence (x_n) such that $x_n \longrightarrow x_0$ and $x_n \in K \cap E_2$. In view of the properties of limit numbers of multifunctions,

$$\mathrm{ls}_n \mathcal{L}(F, x_n) \subseteq \mathcal{L}(F, x_0).$$

Each of the sets $L(F, x_n)$ lies above c, then

$$L(F|_{E_2}, x_0) \cap [c, +\infty) \neq \emptyset.$$

Of course,

$$L(F|_{E_1}, x_0) \cup L(F|_{E_2}, x_0) = L(F, x_0),$$

then

$$c \in L(F|_{E_1}, x_0) \cap L(F|_{E_2}, x_0).$$

$F(x_0) < c$, then the set $L(F|_{E_1}, x_0)$ is nondegenerated. Let now,

$$E_n^{(i)} = \left\{ x \in K : \left(c + \frac{(-1)^i}{n} \right) \in L(F|_{E_i}, x) \right\}$$

for $i = 1, 2$. For each $x, c \notin F(x)$ and $F(x) \cap L(F,x) \neq \emptyset$, hence

$$K = \bigcup_{n=1}^{\infty} (E_n^{(1)} \cup E_n^{(2)}).$$

We infer, from the properties of limit numbers (see [3]), that the sets $E_n^{(i)}$ are closed, thus at least one of them must not be nowhere dense with respect to K, therefore there is $E_n^{(i)}$ that is dense in a portion of K. Let it be $E_n^{(1)}$, and let it be dense in $K \cap [a,b]$ for some $a, b \in \mathbb{R}$, $a < b$. The set $E_n^{(1)}$ is closed, then

$$E_n^{(1)} \cap [a,b] = K \cap [a,b].$$

In that way we have proved that

$$c \in (\inf L(F, x_0), \sup L(F, x_0))$$

and

$$F|_{[x_1,x_2]} \cap (\mathbb{R} \times \{c\}) = \emptyset,$$

what is impossible.

Following the standard way for real functions one can get the analogues of Young's theorems on asymmetry.

Theorem 2 *For any real multifunction F the set*

$$\{x \in \mathbb{R} : \quad F(x) \setminus L^+(F,x) \neq \emptyset\}$$

is denumerable.

Theorem 3 *For any real multifunction F the set*

$$\{x \in \mathbb{R} : \quad L^-(F,x) \neq L^+(F,x)\}$$

is denumerable.

Theorem 4 *The set of all points at which a real multifunction has Darboux property from exactly one side is denumerable.*

Proof. Let F be an arbitrary real multifunction and E be the set of all points of $\mathbb{R}$ at which F has Darboux property from exactly one side. We shall give the proof that the set A of all the points of $\mathbb{R}$ at which F has Darboux property from the left side and has not this property from the right side.

Let A_n be the set of all points x from A for which there is

$$c \in (\inf L^+(F,x), \sup L^+(F,x)),$$

$$\sup L^+(F,x) - \inf L^+(F,x) \geq \frac{1}{n}$$

and

$$c \notin F(t) \quad \text{for all} \quad t \in (x, x+1).$$

One can see that each of the sets A_n contains no right-hand sided accumulation points, hence each A_n is denumerable. Moreover,

$$A = \bigcup_{n=1}^{\infty} A_n,$$

what proves that A is a denumerable set. Similarly, one can prove that the set B of all points of $\mathbb{R}$ at which F has Darboux property from the right side but has not this property from the left side is denumerable, therefore $E = A \cup B$ is also denumerable.

Theorem 5 *The set of all points at which an arbitrary real multifunction has Darboux property is of the type G_δ.*

Proof. Let F be an arbitrary multifunction and E be the complement of the set of all Darboux points of F. Let

$$A = \{x \in E : \quad F(x) \setminus L(F,x) \neq \emptyset\}$$

$$B = \{x \in E : \quad L^-(F,x) \neq L^+(F,x)\}$$

$$C = \{x \in E : \quad F \text{ is Darboux from exactly one side}\}$$

$$D = E \setminus (A \cup B \cup C).$$

In view of theorems 2, 3 and 4, the sets A, B, C are denumerable.

Let C_n denote the set of all points of the set C for which

$$(\exists c \in (\inf L(F,x), \sup L(F,x))) \left(\forall t \in \left(x, x + \frac{1}{3^n}\right)\right) (c \notin F(t)).$$

It is easy to see that

$$\overline{C}_n \subseteq C_{n+1},$$

hence

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \overline{C}_n \text{ and } D = \bigcup_{n=1}^{\infty} C_n.$$

Therefore $E = A \cup B \cup C \cup D$ is of the type of F_σ.

In the end it is worth to add that Theorem 4 can be completed and we can get a full characterization of the set of Darboux points of an arbitrary real multifuntion. Namely, each function is a multifunction too and for them Darboux properties as well global as local ones coincide. Since each set of the type G_δ can be the set of Darboux points of some function (see the article [4]), then the next theorem is a simple corollary of that fact.

Theorem 5' *The set $A \subseteq \mathbb{R}$ is the set of all Darboux points of a real multifunction if and only if it is of the type G_δ.*

Next part of the article will deal with the problems of functionally connected multifunctions.

Definition 3 A real multifunction F is said to be functionally connected iff F is Darboux and each continuous function meeting $F(+)$ and $F(-)$ cuts F as well, where $F(+)$ and $F(-)$ are defined for real multifunctions in the following way:

$$F(+) = \{(x,y): \ y > F(x)\},$$

$$F(-) = \{(x,y): \ y < F(x)\}.$$

Definition 4 A real multifunction F is said to be fuctionally connected at a point $x_0 \in \mathbb{R}$ from the right side iff

(a) $F(x_0)$ is a connected subset of $\mathbb{R}$,

(b) $F(x_0) \cap L^+(F, x_0) \neq \emptyset$,

(c) for each $\delta > 0$ and continuous function $f : [x_0, x_0 + \delta] \longrightarrow \mathbb{R}$ for which $f(x_0) \in (\inf L^+(F, x_0), \sup L^+(F, x_0))$ there is x belonging to $(x_0, x_0 + \delta)$ such that $f(x) \in F(x)$.

In the analogous way we define left-hand sided points of functional connectitivity of a multifunction, and we say that a multifunction is functionally connected at a point if it is so from both sides.

Theorem 6 *A real multifunction is functionally connected if and only if it is functionally connected at each point of* $\mathbb{R}$.

The proof is quite analogous to Theorem 1 and Theorem 1 from the paper [2].

Theorem 7 *The set of all points at which a real multifunction is functionally connected from exactly one side is denumerable.*

Theorem 8 *The set of all points at which an arbitrary real multifunction is functionally connected is of the type* G_δ.

Theorem 8' *The set $A \subseteq \mathbb{R}$ is the set of all funtional connectivity points of a real multifunction if and only if it is of the type G_δ.*

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