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On the Geometrical Properties of Starlike Maps in Banach Spaces

Tadeusz Poreda

1. Introduction.

Let us denote by X the complex Banach space with the norm $\|\cdot\|$. The open ball $\{x \in X : \|x - x_0\| < r\}$ is denoted by $B(x_0, r)$; the unit ball, for short by $B(0,1) = B$. The class of all continuous linear functionals on X regarded as a complex linear space we denote by X'. For each $x \in X$ we define the set

$$T(x) = \{x' \in X' : \|x'\| = 1, \quad x'(x) = \|x\| \}.$$

If Y is another complex Banach space and Ω is the region in X then the function $f : \Omega \longrightarrow Y$ is called (F)-differentiable at the point $x_0 \in \Omega$ when there exists the limit

$$\lim_{\beta \to 0} \frac{1}{\beta}[f(x_0 + \beta h) - f(x_0)] = Df(x_0)(h)$$

for all $h \in X$ and $Df(x_0)$ is the bounded linear operator from X into Y, what means $Df(x_0) \in L(X,Y)$. The norm in $L(X,Y)$ will be

$$\|A\| = \sup\{ \|A(x)\| : \|x\| \leq 1 \} \quad \text{for } A \in L(X,Y).$$

The letter I will always represent the identity map on X. We call the map $f : \Omega \longrightarrow X$ holomorphic on $\Omega \subset X$ when f is (F)-differentiable at all points of Ω. By $H(\Omega)$ we denote the class of holomorphic functions given on Ω with values in X. Then let M be the following subset of $H(B)$:

$$M = \{h \in H(B): \quad h(0) = 0, \quad Dh(0) = I, \quad \operatorname{re} x'(h(x)) > 0$$

$$\text{for } x \in B - \{0\} \text{ and } x' \in T(x)\}.$$

We shall say that the function $f \in H(B)$ is a starlike map if and only if f is one-to-one, $f(0) = 0$ and $(1-t)f(B) \subset f(B)$ for all $t \in [0,1]$.

In this paper we will study the class $\mathcal{G}_0(B)$ of all function $f \in H(B)$ which satisfy the conditions : $f(0) = 0$, $Df(0) = I$, f is locally biholomorphic on B and $f(B)$ is a starlike region in X.

We will consider X with a semi-inner product structure (introduced by Lummer and Philips in [5] and [6]) to obtain the results analogous as in [4]. In a Banach space X we get the semi-inner product as it follows. Let us choose one nonzero element with the norm equal to 1 from each complex line in X containing the point $x = 0$ and denote the set of all chosen elements by X_0. Then, for each $y \in X_0$ let us insert any functional $J_0(y) \in T(y)$. We have defined the map $J_0 : X_0 \longrightarrow X'$. Let us extend that map onto X putting$J(\lambda y) = \overline{\lambda} J_0(y)$ for $y \in X_0$ and $\lambda \in C$ (C denotes the set of complex numbers). Now we can define the semi-inner product denoted by $\langle\cdot,\cdot\rangle$. For $x, y \in X$ we put

$$\langle x, y\rangle = J(y)(x).$$

It has the following properties:

a) it maps $X \times X$ into C,

b) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \langle\lambda x, y\rangle = \lambda\langle x, y\rangle, \quad \langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$ for $x, y \in X$ and $\lambda \in C$,

c) $\langle x, x\rangle = \|x\|^2$ for each $x \in X$

d) $|\langle x, y\rangle|^2 \le \langle x, x\rangle \cdot \langle y, y\rangle$ for $x, y \in X$.

2. The parametric representation of starlike maps of the unit ball in Banach spaces.

Lemma 1 *Let $h \in M$. Then for each $x \in B$ the initial value problem*

$$\frac{\partial v}{\partial t}(x,t) = -h(v(x,t)), \quad v(x,0) = x \tag{1}$$

has the unique solution $v(x,t)$ which is defined for all $t \geq 0$. Furthermore, for all these t the function $v_t(x) = v(x,t)$ is the univalent Schwarz function on B which has its derivatives for all orders with respect to the pair of variables $(x,t) \in B \times [0,\infty)$.

The following inequalities hold

$$\begin{cases} \frac{\|v(x,t)\|}{(1+\|v(x,t)\|)^2} \geq e^{-t}\frac{\|x\|}{(1+\|x\|)^2} \\ \frac{\|v(x,t)\|}{(1-\|v(x,t)\|)^2} \leq e^{-t}\frac{\|x\|}{(1-\|x\|)^2} \end{cases} \tag{2}$$

for all $x \in B$ and $t \geq 0$.

Proof. The existence and uniqueness of the solution of the problem (1) follow from Lemma 5 in the paper [3]. From that paper we have as well that the function $v_t(x) = v(x,t)$ is the univalent Schwarz function on B for every fixed $t \geq 0$. By applying Theorem IX 5′ from [7] to the function v the existence of its derivatives of all orders with respect to $(x,t) \in B \times [0,\infty)$ can easily be proved. Hence we have to prove only the inequalities (2). Lemma 5 in [3] implies that for almost every $t \geq 0$ we have

$$\|v(x,t)\|\frac{\partial\|v(x,t)\|}{\partial t} = \mathrm{re}\left\langle\frac{\partial v(x,t)}{\partial t}, v(x,t)\right\rangle$$

Using the ineqality (24) from [3] and $Dh(0) = I$ we obtain for all $x \in B$ and almost all $t \in [0,\infty)$

$$\mathrm{re}\,\langle h(v(x,t)), v(x,t)\rangle \geq \frac{1-\|v(x,t)\|}{1+\|v(x,t)\|}\|v(x,t)\|^2.$$

Hence

$$\frac{\partial\|v(x,t)\|}{\partial t} \leq -\frac{1-\|v(x,t)\|}{1+\|v(x,t)\|}\|v(x,t)\| \tag{3}$$

for $x \in B$ and a.e. on $[0,\infty)$.

Since $\|v(x,t)\|$ is an absolutely continuous function with respect to the parameter t, then by integrating each side of the inequality

$$\frac{1+\|v(x,t)\|}{(1-\|v(x,t)\|)\|v(x,t)\|}\cdot\frac{\partial\|v(x,t)\|}{\partial t}\leq -1$$

on the interval $[0,t]$ we obtain

$$\frac{\|v(x,t)\|}{(1-\|v(x,t)\|)^2}\leq e^{-t}\frac{\|x\|}{(1-\|x\|)^2}$$

for $x \in B$ and $t \geq 0$.

The proof of the second inequality of (2) is analogous.

Lemma 2 *If $h \in M$ then*

$$|\langle\frac{1}{2!}D^2h(0)(x,x),x\rangle|\leq 2$$

for $x \in B$.

Proof. Let $x \in B$ and $x \neq 0$. Considering the construction of semi-inner product we notice that if $h \in M$ then the function

$$p(\lambda)=\begin{cases}\frac{1}{\lambda\|x\|}\langle h(\frac{x}{\|x\|}\lambda),x\rangle & \text{for} \quad 0<|\lambda|<1\\ 1 & \text{for} \quad \lambda=0\end{cases}$$

is holomorphic for $|\lambda|<1$, and re $p(\lambda)>0$ for $|\lambda|<1$. Thus

$$|p'(0)|\leq 2.$$

It can also be shown that

$$p'(0)=\frac{\langle\frac{1}{2!}D^2h(0)(x,x),x\rangle}{\|x\|^3}.$$

Then

$$|\langle \frac{1}{2!} D^2 h(0)(x,x), x\rangle| \leq 2 \tag{4}$$

which ends the proof.

Theorem 1 *Let $h \in M$ and for $x \in B$ and $t \geq 0. v(x,t)$ be the solution of the problem (1) and $\frac{\partial v}{\partial x}(x,t)$ be an invertible linear operator. Then v satisfies the differential equation*

$$\frac{\partial v}{\partial x}(x,t) h(x) = -\frac{\partial v}{\partial t}(x,t) \tag{5}$$

for $x \in B$ and $t \geq 0$.

Proof. The proof of Theorem 1 from [4] can be strictly repeated in our case.

Theorem 2 *If a function $v = v(x,t)$ for $x \in B$ and $t \geq 0$ satisfies the equation (1) with any fixed $h \in M$, then for all $x \in B$ there exists the limit*

$$\lim_{t \to \infty} e^t v(x,t) = f(x) \tag{6}$$

and the function f is holomorphic on B.

Proof. We denote $u(x,t) = e^t v(x,t)$ for $x \in B$ and $t \geq 0$. Thus u fulfils the equation

$$\frac{\partial u}{\partial t}(x,t) = u(x,t) - e^t h(e^{-t} u(x,t)) , \quad u(x,0) = x. \tag{7}$$

For $x \in B$ we will denote $G(x) = h(x) - x. G$ is holomorphic on B, $G(0) = 0$ and $DG(0) = 0$. Using the new notation we can rewrite (7) in the form

$$\frac{\partial u}{\partial t}(x,t) = -e^t G(e^{-t} u(x,t)), \quad u(x,0) = x \tag{7'}$$

for $x \in B$ and $t \geq 0$.
Now after we integrated the equation (7′) on the interval $[t_1, t_2]$, where $0 < t_1 < t_2$ we obtain

$$u(z, t_2) - u(z, t_1) = -\int_{t_1}^{t_2} e^{\tau} G(e^{-\tau} u(z, \tau)) d\tau. \tag{8}$$

For G is holomorphic on B so it is also locally bounded on B. Hence there exists such a ball $B(0, r) \subset B$ that for all $x \in B(0, r)$ we have $\|G(x)\| \leq K$ where K is a positive constant from Cauchy integral formula (see [2] p.101) there exists another ball $B(0, \delta) \subset B(0, \frac{r}{2})$ such that for all $a, x \in B(0, \delta)$ the following equality holds

$$\frac{1}{2!} D^2 G(a)(x, x) = \frac{1}{2\pi i} \int_{|t|=1} f(a + tx) t^{-3} dt \tag{9}$$

Since $a, x \in B(0, \delta)$ then $a + tx \in B(0, r)$ when $|t| = 1$. Combining this fact with (9) we will get

$$\|\frac{1}{2!} D^2 G(a)(x, x)\| \leq K$$

for $a, x \in B(0, \delta)$. The second derivative of G at the point $a \in B$ is a bilinear operator, then

$$\|\frac{1}{2!} D^2 G(a)(y, y)\| \leq \frac{K\|y\|^2}{\delta^2} \tag{10}$$

when $a \in B(0, \delta)$ and $y \in X$.

Now we use the Taylor formula and inequality (10) to obtain

$$\|G(y)\| \leq \frac{K\|y\|^2}{\delta^2} \tag{11}$$

for $y \in B(0, \delta)$.
On account of (2), for each $r \in (0, 1)$ there exists some $\tau_r \geq 0$ such that, for $\tau > \tau_r$ and $\|x\| < r$

$$\frac{u(x, \tau)}{\tau} \in B(0, \delta).$$

Using (8) and (11) we establish

$$\|u(x,t_2)-u(x,t_1)\| \le K\int_{t_1}^{t_2}\tau^2 e^{-\tau}d\tau. \tag{12}$$

for $\|x\|<r$ and $t_1,t_2>\tau_r$.

The function $g(\tau)=\tau^2e^{-\tau}$ is integrable on the interval $(0,\infty)$. This implies that for every $\varepsilon>0$ there exists $\tau_r^{'}>0$ such that for $t_1,t_2>\tau_r^{'}$

$$\sup_{\|x\|<r}\|u(x,t_1)-u(x,t_2)\|<\varepsilon.$$

Hence, the Weierstrass theorem (see [2], proposition 6.5) and completeness of X yield that there exists $\lim_{t\to\infty}u(x,t)$ for all $x\in B$ and it forms the holomorphic function on B. It ends the proof.

Corollary 1 *If $v=v(x,t)$ for $x\in B$ and $t\ge 0$ fulfils the initial value problem (1) with some $h\in M$, then*

$$\lim_{t\to\infty}(-e^{-t}\,\frac{\partial v}{\partial t}(x,t))=\lim_{t\to\infty}e^tv(x,t) \tag{13}$$

for all $x\in B$.

Proof. We will use the notations as in Theorem 2. It remains to show that for all $x\in B$

$$\lim_{t\to\infty}\frac{\partial u}{\partial t}(x,t)=0.$$

Considering $(7^{'})$ we can remark that for $t>\tau_r$ and $\|x\|<r$

$$\|\frac{\partial u}{\partial t}(x,t)\|\le Kt^2e^{-t}.$$

It implies that $\lim_{t\to\infty}\frac{\partial u}{\partial t}(x,t)=0$ for $x\in B$ what completes the proof.

Lemma 1 lets us to prove, similarly as in [4], the following theorem.

Theorem 3 *Let $h\in M$ and $v=v(x,t)$ for $x\in B$ and $t\ge 0$ be the solution of (1). Then the limit*

$$\lim_{t\to\infty}e^tv(x,t)=f(x)$$

is a starlike function on B such that $f(0)=0$, $Df(0)=I$ and $f(v(x,t))=e^{-t}f(x)$ for $x\in B$ and $t\ge 0$.

Theorem 4 *Let $h \in M$ and $v = v(x,t)$ for $x \in B$ and $t \geq 0$ be a solution of equation (1). If a map f defined by the equality*

$$f(x) = \lim_{t \to \infty} e^t v(x,t) \quad \text{for } x \in B$$

is locally biholomorphic, then it satisfies the equation

$$Df(x)h(x) = f(x), \quad \text{for} \quad x \in B.$$

Furthermore, f is a unique locally biholomorphic solution of this equation such that $f(0) = 0$, $Df(0) = I$.

Proof. Let v and f satisfy the assumption our theorem. From Theorem 3 we infer that

$$f(v(x,t)) = e^{-t} f(x), \quad \text{for } x \in B \quad \text{and} \quad t \geq 0.$$

This equality implies that

$$Df(v(x,t)) \frac{\partial v}{\partial x}(x,t) = e^{-t} Df(x), \quad \text{for} \quad x \in B \ \text{and} \ t \geq 0.$$

Since f is a locally biholomorphic map then, in virtue of the above equality, we obtain that $\frac{\partial v}{\partial x}(x,t)$ is an invertible linear operator for $x \in B$ and $t \geq 0$. A continuation of the proof of this theorem runs similarly as that of Theorem 4 from [4].

Theorem 5 *If $f \in \mathcal{G}_0(B)$, then*

$$f(x) = \lim_{t \to \infty} e^t v(x,t),$$

where $v(x,t)$, for $x \in B$ and $t \geq 0$ is a solution of equation (1) with function $h(x) = (Df(x))^{-1} f(x)$ for $x \in B$

Proof. From the assumption it follows that $(Df)^{-1} \circ f \in M$. So by Theorem 2 for all $x \in B$ there exists the limit $\lim_{t \to \infty} e^t v(x,t)$. Since $h = (Df)^{-1} \circ f$ and v satisfies the equation (1), we have

$$\frac{\partial f(v(x,t))}{\partial t} = -f(v(x,t)) \quad \text{and} \quad v(x,0) = x$$

for $x \in B$ and $t \geq 0$. Integrating this equation we get

$$f(v(x,t)) = e^{-t} f(x) \quad \text{for} \quad x \in B \quad \text{and} \quad t \geq 0.$$

Hence

$$v(x,t) = f^{-1}(e^{-t} f(x)) \quad \text{for} \quad x \in B \quad \text{and} \quad t \geq 0.$$

It is not difficult to show that $\lim_{t\to\infty} e^t v(x,t) = f(x)$ for $x \in B$.

3. The geometrical properties of starlike maps of the unit ball in Banach spaces.

Theorem 6 *If $f \in \mathcal{G}_0(B)$, then the following inequalities hold*

$$\frac{\|x\|}{(1+\|x\|)^2} \leq \|f(x)\| \leq \frac{\|x\|}{(1-\|x\|)^2} \tag{14}$$

for all $x \in B$.

Proof. Since $f \in \mathcal{G}_0(B)$ then from Theorem 7 in [3], there exists such a function $h \in M$ that f satisfies the equation

$$Df(x)h(x) = f(x) \quad \text{for all} \quad x \in B.$$

Theorem 5 shows that

$$f(x) = \lim_{t\to\infty} e^t v(x,t) \quad \text{for} \quad x \in B,$$

where $v(x,t)$ fulfils (with the given function h) the equation (1). Hence

$$\|f(x)\| = \lim_{t\to\infty} e^t \|v(x,t)\| \quad \text{for} \quad x \in B.$$

The inequalities (2) take place for the function v, so

$$\lim_{t\to\infty} \|v(x,t)\| = 0$$

for $x \in B$ and consequently

$$\frac{\|x\|}{(1+\|x\|)^2} \leq \|f(x)\| \leq \frac{\|x\|}{(1-\|x\|)^2}$$

for all $x \in B$.

Theorem 7 *If $f \in \mathcal{G}_0(B)$, then for all $x \in B$*

$$\|(Df(x))^{-1}\| \geq \frac{(1-\|x\|)^3}{1+\|x\|} \tag{15}$$

Proof. Let $f \in \mathcal{G}_0(B)$. There exists $h \in M$ such that f satisfies the equality

$$Df(x)h(x) = f(x) \quad \text{for} \quad x \in B.$$

(see Th. 7 from [3]). Next we can remark that

$$\langle h(x), x\rangle = \langle (Df(x))^{-1} f(x), x\rangle$$

for $x \in B$ and, considering (24) from [3], that

$$\operatorname{re}\langle h(x), x\rangle \geq \|x\|^2 \frac{1-\|x\|}{1+\|x\|} \quad \text{for} \quad x \in B. \tag{16}$$

Using the properties c) and d) of the semi-inner product we obtain

$$|\langle h(x), x\rangle| \leq \|(Df(x))^{-1} f(x)\| \cdot \|x\| \quad \text{for } x \in B. \tag{16'}$$

The properties of the linear operator's norm and the inequalities (14) give us

$$\|(Df(x))^{-1} f(x)\| \leq \|(Df(x))^{-1}\| \frac{\|x\|^2}{(1-\|x\|)^2} \quad \text{for } x \in B$$

and further, applying (16) and (16')

$$\|x\|^2 \frac{1-\|x\|}{1+\|x\|} \leq \|(Df(x))^{-1}\| \cdot \frac{\|x\|^2}{(1-\|x\|)^2} \quad \text{for } x \in B.$$

It gives (15).

Theorem 8 *If $f \in \mathcal{G}_0(B)$, then*

$$|\langle \frac{1}{2!} D^2 f(0)(x,x), x\rangle| \leq 2 \quad \textit{for} \quad x \in B. \tag{17}$$

Proof. The paper [3] shows the existence of such a function $h \in M$ that $Df(x)\,h(x) = f(x)$ for $x \in B$. Let us fix $x_0 \in B$. We will consider two functions

$$G_{x_0}(\lambda) = Df(\lambda x_0)h(\lambda x_0),$$

$$H_{x_0}(\lambda) = f(\lambda x_0)$$

for $|\lambda| < 1$. They are holomorphic on the unit ball in C and map it into the Banach space X. Naturally $G_{x_0} = H_{x_0}$.
It is easy to show that

$$G_{x_0}(0) = 2D^2 f(0)(x_0, x_0) + D^2 h(0)(x_0, x_0)$$

and

$$H_{x_0}(0) = D^2 f(0)(x_0, x_0)$$

Hence

$$D^2 f(0)(x_0, x_0) + D^2 h(0)(x_0, x_0) = 0.$$

Taking an arbitrary $x \in B$ we can obtain

$$D^2 f(0)(x, x) = -D^2 h(0)(x, x).$$

Now one should apply lemma 2 to get (17).

Theorem 9 *If $f \in \mathcal{G}_0(B)$, then for each $n \in N$ and $n \geq 2$*

$$\|\frac{1}{n!} D^{(n)} f(0)(x^n)\| \leq \frac{e^2}{4}(n+1)^2 \quad \text{for all } x \in B. \tag{18}$$

Proof. Let x' be the fixed functional from X' such that $\|x'\| = 1$ and let $x_0 \in B$. We define the function F_{x_0} in the following way

$$F_{x_0}(\lambda) = x'(f(\lambda x_0)) \quad \text{for } |\lambda| < 1.$$

This function is holomorphic on the unit ball in C and with regard to theorem 6, satisfies the condition

$$|F_{x_0}(\lambda)| \leq \frac{|\lambda|\|x_0\|}{1 - |\lambda|\|x_0\|} \quad \text{for } |\lambda| < 1.$$

Applying the Cauchy inequality to

$$F_{x_0}^{(n)}(0) = x'(D^{(n)}f(0)(x_0^n)) \quad \text{for } n \in N,$$

we get

$$|\frac{1}{n!}x'(D^{(n)}f(0)(x_0^n))| \leq \frac{\|x_0\|}{r^{n-1}(1-r\|x_0\|)^2}$$

for each $0 < r < 1$ and $n \in N$, $n \geq 2$. Since

$$\min_{0<r<1} \frac{\|x_0\|}{r^{n-1}(1-r\|x_0\|)^2} \leq \frac{e^2}{4}(n+1)^2 \quad \text{for } n \geq 2,$$

then

$$(19) \qquad |\frac{1}{n!}x'(D^{(n)}f(0)(x_0^n))| \leq \frac{e^2}{4}(n+1)^2 \quad \text{for } n \geq 2.$$

Using Theorem 8.3 from [1] p.139, for any fixed $x_0 \in B$ and $n \geq 0$ we conclude that there exists such a functional $x_0' \in X'$ that $\|x_0'\| = 1$ and

$$x_0'(D^{(n)}f(0)(x_0^n)) = \|D^{(n)}f(0)(x_0^n)\|.$$

Now, applying (19) we obtain (18).

Remark In the case $X = C$ the results of Theorems 6,7 and 8 assume the form of the well know estimations for starlike functions of one complex variable. So when $X = C$ the inequality (14) gives us

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \quad \text{for } |z| < 1,$$

from (15) we get

$$|f(z)| \leq \frac{1+|z|}{(1-|z|)^3} \quad \text{for } |z| < 1$$

and from (17)

$$|\frac{1}{2!}f''(0)| \leq 2.$$

This fact implies that we cannot improve obtained results generally for an arbitrary Banach space.

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POLITECHNIKA ŁÓDZKA
INSTYTUT MATEMATYKI
Al. Politechniki 11
93 590 Łódź, Poland