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Some Differential Inequalities

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1. Preliminaries Let us establish some terminology to be used. $\mathbb{R}$ denotes the real line and $\mathbb{N}$ denotes the set of all positive integers. ∂A and $\overline{A}$ denote the boundary and closure of a set A, respectively. $\mathbb{R}^n$ denotes a normed, real n-dimensional vector space of elements $y = (y^1, \ldots, y^n)$ with a norm $| y |$. Unless otherwise specified, $| y |$ will be the norm defined by $| y |= \max(| y^1 |, \ldots, | y^n |)$ and $\| y \|$ will denote the Euclidean norm.

A family $\mathcal{F}$ of functions $f : E \longrightarrow Y$, where $\emptyset \neq E \subseteq X$, $(X, | \cdot |_x)$ and $(Y, | \cdot |_y)$ are normed spaces, is called to be equicontinuous if for every $\varepsilon > 0$ there exists a $\delta = \delta_\varepsilon > 0$ such that $| f(y_1) - f(y_2) |_Y \leq \varepsilon$ whenever $y_1, y_2 \in E$, $| y_1 - y_2 |_X \leq \delta$ and $f \in \mathcal{F}$.

Let $A \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. For $x \in \mathbb{R}^n$ we will define the upper density (the lower density) of A at a point x by the upper limit (the lower limit) of the set of all numbers of the form

$$\limsup \frac{\mu(A \cap I_t)}{\mu(I_t)} \qquad \left(\text{suitably } \liminf \frac{\mu(A \cap I_t)}{\mu(I_t)}\right)$$

for all sequences of intervals $\{I_t\}_{t \in T}$, such that $x \in I_t$ and the diameter of I_t tending to zero. ($\mu(A)$ denotes n-dimensional Lebesgue measure of A). If the upper density and the lower density are equal 1 at a point x we say that x is a point of density of A.

A measurable Lebesgue function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is integrable on every interval is approximately continuous at a point x if for every open set $U \subseteq \mathbb{R}$ such that $f(x) \in U$, x is a point of density of set $f^{-1}(U)$.

Let $\mathcal{P}$ be a family of all intervals contained in $\mathbb{R}^n$. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ integrable with respect to Lebesgue measure μ on every interval $I \in \mathcal{P}$ is a derivative if

$$\lim_{I \Rightarrow x,\ I \in \mathcal{P}} \frac{\int_I f(t)d\mu}{\mu(I)} = f(x)$$

(here the symbol $I \Rightarrow x$ means that $x \in I$ and the diameter of I tends to zero).

Let $f : E \longrightarrow \mathbb{R}^n$, (where

$$\emptyset \neq E = \{(t, y) \in \mathbb{R}^{n+1} : \ t \in A \subseteq \mathbb{R}, \ \ y \in B \subseteq \mathbb{R}^n\})$$

be a function such that

(i) the family $\{f_t(v) = f(t, v)\}_{t \in A}$ is equicontinuous,

(ii) $f^y = f(v, y)$ is a locally bounded derivative for every $y \in B$.

The functions fulfilling the conditions (i) and (ii) we shall call functions with $\mathcal{F}$ property (or simply $\mathcal{F}$-functions).

2. Integral inequality

We shall consider the generalization of Gronwall inequality. This integral inequality is reducible to differential inequality.

Theorem 1 *Let u, v be two real non-negative functions defined on an interval $[t_0, t_0 + a]$. Assume that u is an approximately continuous function and v is a derivative. Let C be a non-negative real number. Suppose further that the condition*

(2.1) $\quad v(t) \leq C + \int_{t_0}^t v(s)u(s)ds \quad$ *for* $\ t \in [t_0, t_0 + a)$

holds.

Then

(2.2) $\quad v(t) \leq C \exp(\int_{t_0}^t u(s)ds) \quad$ *for* $\ t \in [t_0, t_0 + a)$.

In particular, if $C = 0$ then $v = 0$.

Proof. We shall consider two possibilities:

$$(1) \quad C > 0,$$

$$(2) \quad C = 0.$$

If (1), then from (2.1) we infer the following condition

$$\frac{v(t)u(t)}{C + \int_{t_0}^{t} v(s)u(s)ds} \leq u(t).$$

Because $u \cdot v$ is a bounded derivative (by Iosifescu's Theorem, see [3]), therefore by integration we get

$$\log(C + \int_{t_0}^{t} v(s)u(s)ds) - \log C \leq \int_{t_0}^{t} u(s)ds.$$

This inequality, together with (2.1) imply (2.2). If (1), then there exists a sequence $\{c_n\}_{n \in \mathbb{N}}$ such that $c_n > 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = 0$. Since (1) implies (2.2) for every $n \in \mathbb{N}$, then

$$v(t) \leq \lim_{n\to\infty} c_n \exp(\int_{t_0}^{t} u(s)ds) = 0.$$

Since v is a non-negative function, then $v = 0$ on the interval $[t_0, t_0 + a)$. The proof is completed.

3. Theorems for maximal and minimal solutions

Let $f : E \longrightarrow \mathbb{R}^n$, where E is a nonempty subset of $\mathbb{R}^{n+1}$. Let u be a solution of the equation

$$(3.1) \quad u' = f(t, u)$$

satisfying the condition $u(t_0) = w_0$, $(t_0, w_0) \in E$ on an interval J.An interval K is called a right maximal interval of existence of the solution u of (3.1) if there exists no extension of u on any interval K_1 such that $K \subseteq K_1$, $K \neq K_1$, K_1 and K have different right end-points so that u remains a solutions of (3.1). Left maximal interval of existence for u is defined similarly. A maximal interval of existence is an interval which is both a left and right maximal interval of existence.

In [4] Lakshmikantham and Leela proved the following.

Lemma 1 *Let v, w be two real non-negative, continuous functions defined on $[t_0, t_0 + a)$, D be some Dini derivative. If*

(3.2) $Dv(t) \leq w(t)$ *on* $[t_0, t_0 + a)$

except of a countable set S, then $D_-v(t) \leq w(t)$ for $[t_0, t_0 + a)$, where D_-v denotes a left lower Dini derivative of v.

We shall show a fundamental result on scalar differential inequalities.

Theorem 2 *Assume that:*

(3.3) *E is a nonempty open subset of* $\mathbb{R}^2$,

(3.4) $g : E \longrightarrow \mathbb{R}$,

(3.5) $v : [t_0, t_0 + a] \longrightarrow \mathbb{R}$ *and* $w : [t_0, t_0 + a] \longrightarrow \mathbb{R}$ *are continuous for some* $a > 0$, $(t, v(t)) \in E$, $(t, w(t)) \in E$, $t \in [t_0, t_0 + a]$

and

(3.6) $v(t_0) < w(t_0)$,

(3.7) $D_-v(t) \leq g(t, v(t))$ *for* $t \in (t_0, t_0 + a)$,

(3.8) $D_-w(t) > g(t, w(t))$ *for* $t \in (t_0, t_0 + a)$.

Then

(3.9) $v < w$ *on* $[t_0, t_0 + a)$.

Proof. Suppose that (3.9) is false. Then the set

$$Z_1 = \{t \in [t_0, t_0 + a), \quad w(t) \leq v(t)\} \neq \emptyset.$$

Let $t_1 = \inf Z_1$. It is clear (from (3.6)) that $t_0 < t_1$. Moreover

(3.10) $v(t_1) = w(t_1)$

and

(3.11) $\quad v(t) < w(t) \quad \text{for} \quad t \in [t_0, t_1).$

From (3.7) and (3.8), we obtain

$$\frac{v(t_1-h)-v(t_1)}{-h} > \frac{w(t_1-h)-w(t_1)}{-h}$$

for sufficiently small $h > 0$. Thus

$$D_-v(t_1) \geq D_-w(t_1).$$

Applying the conditions (3.7), (3.8), (3.9) and (3.10) we obtain a contradiction to $g(t_1, v(t_1)) > g(t_1, w(t_1))$. Hence $Z_1 = \emptyset$ and the proof is completed.

Remark 1 *It is clear that the inequalities (3.7) and (3.8) can be replaced by*

$$D_-v(t) < g(t, v(t)), \qquad D_-w(t) \geq g(t, w(t)),$$

respectively.

Theorem 3 *Let g, E, v, w fulfil the assumptions of Theorem 3.1 for $t \in Z_1 = \{t \in (t_0, t_0 + a), v(t) = w(t)\}$. Then (3.9) holds.*

Proof. Since we needed that in the proof of Theorem 3.1 the inequalities were satisfied for $t \in Z_1$ so this theorem has a similar proof.

Remark 2 *From Lemma 3.1 we obtain that Theorems 3.1 and 3.2 are true, when the inequalities (3.7) and (3.8) hold for $t \in [t_0, t_0 + a) \setminus S$ and D is a fixed Dini derivative.*

Lemma 2 *Let $F : A \longrightarrow \mathbb{R}$ be a function with $\mathcal{F}$ property, where $A \subseteq \mathbb{R}^2$ is a rectangle defined by $t_0 \leq t \leq t_0 + a$, $\mid u - w_0 \mid \leq b$, $(a, b > 0)$. Let M denote a positive number such that $\mid F \mid \leq M$ for $(t, u) \in A$.*

Then there exist a maximal solution and a minimal solution of

(3.12) $\quad u' = g(t, u),\ u(t_0) = u_0 \text{ on } [t_0, t_0+\alpha], \quad \text{where } \alpha = \min\{a, \frac{b}{2M+b}\}.$

Proof. Note that the problem of existence of the minimal solution is similar to the existence of the maximal solution, therefore we shall show the theorem only for the maximal solution. Consider the initial value problem

$$(3.13)\qquad u' = g(t,u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon, \quad \text{where } 0 < \varepsilon \leq \tfrac{b}{2}.$$

Observe that $g_\varepsilon(t,u) = g(t,u) + \varepsilon$ is well defined function and has $\mathcal{F}$ property on R_ε:

$$t_0 \leq t \leq t_0 + a, \ \ | u - (u_0 + \varepsilon) | \leq \frac{b}{2}, \quad R_\varepsilon \subseteq R_0 \text{ and } \ | g_\varepsilon | \leq M + \frac{b}{2} \text{ on } R_\varepsilon.$$

From Peano Existence Theorem [1] we obtain that the differential equation (3.13) has a solution $u(.,\varepsilon)$ on the interval $[t_0, t_0 + \alpha]$, where $\alpha = \min\{a, \frac{b}{2M+b}\}$. Let $0 < \varepsilon_2 < \varepsilon_1 \leq \varepsilon$. Then we have

$$u(t_0,\varepsilon_2) < u(t_0,\varepsilon_1),$$

$$u'(t_0,\varepsilon_2) \leq g(t, u(t_0,\varepsilon_2)) + \varepsilon_2,$$

$$u'(t,\varepsilon_1) > g(t, u(t,\varepsilon_1)) + \varepsilon_1 \quad \text{on} \quad [t_0, t_0 + \alpha].$$

From Theorem 3.1 we obtain

$$u(t,\varepsilon_1) > u(t,\varepsilon_2) \quad \text{for} \quad t \in [t_0, t_0 + \alpha].$$

Note that the family of functions $\{u(t,\varepsilon_n)\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded on $[t_0, t_0 + \alpha]$, so from Ascoli-Arzela Theorem [4] it follows that there is a decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ and } z(t) = \lim_{n\to\infty} u(t,\varepsilon_n)$$

exist for every $t \in [t_0, t_0 + \alpha]$. Of course, $z(t_0) = u_0$. Since g is $\mathcal{F}$-function, then $g(t, u(t,\varepsilon_n))$ tends uniformly to $g(t, z(t))$ if $t \longrightarrow \infty$. Thus we can apply Lebesgue Theorem [5,Th.20, page 321] to $u(t,\varepsilon_n) = u_0 + \int_{t_0}^t g(s, u(s,\varepsilon_n))ds$, which in turn shows that

$$z(t) = u_0 + \int_{t_0}^t g(s, z(s))ds$$

and z is a solution of (3.12), then $u(t) \le z(t)$ for every $t \in [t_0, t_0 + \alpha]$. Let u be a solution of (3.12) which is defined on $[t_0, t_0 + \alpha]$. Then

$$u(t_0) = u_0 < u_0 + \varepsilon = u(t_0, \varepsilon),$$

$$u'(t) < g(t, u(t)) + \varepsilon,$$

$u'(t,\varepsilon) \ge g(t, u(t,\varepsilon)) + \varepsilon$ on the interval $[t_0, t_0 + \alpha]$ and for $\varepsilon \le \frac{b}{2}$.

By Remark 3.1 we have $u(t) < u(t, \varepsilon)$ for $t \in [t_0, t_0 + \alpha]$. Thus $u < z$ for every $t \in [t_0, t_0 + \alpha]$. This completes the proof.

Theorem 4 *Let $F : E \longrightarrow \mathbb{R}$ be a function with $\mathcal{F}$ property, where E ($\emptyset \neq E \subseteq \mathbb{R}^2$) is an open set, and $(t_0, w_0) \in E$. Then (3.9) has a maximal and minimal solution.*

Proof. By Lemma 3.2, the equation (3.12) has a maximal solution u^* and a minimal solution u_* on $[t_0, t_0 + \alpha]$. By Theorem 6 [2], an arbitrary solution can be extended onto a maximal interval of existence (ω_-, ω_+), so u_*, u^*tends to the boundary ∂E of E, when $t \longrightarrow \omega_-$ and $t \longrightarrow \omega_+$.

4. Differential Inequalities

For functions satisfying the initial value problem

$$u' = F(t, u), \; u(t_0) = u_0$$

some estimations by extremal solutions are considered. The first of those theorems is one of the results applied quite often in the theory of differential equations.

Theorem 5 *Assume that*

(4.1) *$F : E \longrightarrow \mathbb{R}$ is a bounded $\mathcal{F}$-function, where E is a nonempty subset of $\mathbb{R}^2$,*

(4.2) *$u = u^*$ is a maximal solution of*

(4.3) $u' = F(t, u), \;\; u(t_0) = u_0,$

(4.4) $v : \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function on* $[t_0, t_0 + a]$ *satisfying the inequality* $v(t_0) \leq u_0$ *and* $(t, v(t)) \in E$,

(4.5) $D_{-}v \leq F(t, v)$ *on* $[t_0, t_0 + a]$, *where* $D_{-}v$ *is a lower right Dini derivative.*

Then

(4.6) $v \leq u^*$ *on a common interval of existence of* u^* *and* v.

Proof. Let ε be a positive number less than $\frac{b}{2}$ and

(4.7) $u' = F(t, u) + \varepsilon, \ \ u(t_0) = u_0 + \varepsilon.$

Similarly as in the proof of Lemma 3.2 we can define the function $F_\varepsilon(t, u) = F(t, u) + \varepsilon$ and the rectangle R_ε defined by inequalities $t_0 \leq t \leq t_0 + a, \mid u - (u_0 + \varepsilon) \mid \leq \frac{b}{2}$, satisfying the conditions of Peano Existence Theorem [1]. Thus we obtain that the initial value problem (4.7) has a solution $u(., \varepsilon)$ on the interval $[t_0, t_0 + \alpha]$, where $\alpha = \min\{a, \frac{b}{2M+b}\}$. Similarly as in the proof of Lemma 3.2 we can infer that the function $u^*(t) = \lim_{n\to\infty} u(t, \varepsilon_n)$ is a solution of (4.3), where $\lim_{n\to\infty} \varepsilon_n = 0$ and $u(t, \varepsilon_n)$ is a solution of (4.7) for every $n \in \mathbb{N}$. We shall show that such defined solution u^* satisfies (4.6). Let $n \in \mathbb{N}$. Note that $v(t_0) \leq u_0 < u_0 + \varepsilon_n = u(t_0 + \varepsilon_n)$, from (4.5) it follows that $Dv(t) \leq F(t, v) < F(t, v) + \varepsilon_n = F_{\varepsilon_n}(t, v)$ on the interval $[t_0, t_0 + a]$. Since from (4.7) it follows that

$$Du(t, \varepsilon_n) \geq F(t, u) + \varepsilon_n = F_{\varepsilon_n}(t, u(t, \varepsilon_n)) \quad \text{on} \quad [t_0, t_0 + \alpha]$$

so Lemma 3.1 implies that those inequalities hold for a left lower Dini derivative. Thus $v < u(t, \varepsilon_n)$ on $[t_0, t_0 + \alpha)$ is implied by Theorem 3.1. Since $n \in \mathbb{N}$ is arbitrary then $v \leq u^*$. Thus the proof is completed.

The proofs of next two theorems are very similar to the proof of Theorem 4.1.

Theorem 6 *Let* E, F *fulfil the assumptions of Theorem* 4.1. *Let* u_* *be a minimal solution of* (4.3) *and* $v : [t_0, t_0 + \delta] \longrightarrow \mathbb{R}$ *be a continuous function such that* $v(t_0) \geq w_0$ *and* $(t, v(t)) \in E$. *If some Dini derivative* Dv *satisfies the condition*

(4.8) $\quad Dv \geq F(t,v) \quad on \quad [t_0, t_0 + a],$

then

(4.9) $\quad v \geq u \quad$ *on a common interval of existence of* v *and* u_0.

Theorem 7 *Let* F, E, u^* *fulfil the assumptions of Theorem* 4.1.
If $v : [t_0, t_0 + a] \longrightarrow \mathbb{R}$ *is a continuous function fulfilling the conditions:*

$$v(t_0) \leq w_0, \quad (t, v(t)) \in E$$

and

(4.10) $\quad Dv \leq F(t,v) \quad on \quad [t_0 - a, t_0],$

then

(4.11) $\quad v \leq u^*$ *on a common interval of existence of* u^0 *and* v.

Remark 3 *If* $Dv \geq F(t.v)$ *[or* $Dv \geq F(t,v)$*] on* $[t_0 - a, t_0]$ *and* $v(t_0) \geq w_0$, *then* $v \geq u_0$ *on* $(t_0 - \alpha, t_0]$.

The next corollary is a generalization of Theorem 2.1.

Corollary 1 *Let* $F : E \longrightarrow \mathbb{R}$ *be a nondecreasing function with respect to* u *for* $t_0 \leq t \leq t_0 + a$ *with* $\mathcal{F}$*-property. Let the maximal solution* u^* *of* (4.1) *exist on* $[t_0, t_0 + a]$, *and* $v : [t_0, t_0 + a] \longrightarrow \mathbb{R}$ *be a continuous function satisfying*

(4.12) $\quad v(t) \leq z_0 + \int_{t_0}^{t} F(s, v(s))ds,$ *where* $z_0 \leq w_0$.

Then $v \leq u^*$ *on* $[t_0, t_0 + a]$.

Proof. Let $W(t)$ denote the right part of inequality (4.12). Then $v \leq W$ on $[t_0, t_0 + a]$ and $W'(t) = F(t, v(t))$. By the monotonicity of F, $W'(t) = F(t, v(t)) \leq F(t, W(t))$. Because of $z_0 \leq w_0$ Theorem 4.1 implies that $W \leq u^*$ on $[t_0, t_0 + a]$. Thus $v \leq u^0$ on $[t_0, t_0 + a]$.

5. Generalization of Kamke's uniqueness theorem

One of the principal applications of Theorem 4.1 and its corollary is to obtain uniqueness theorem. First, we shall prove the next proposition.

Lemma 3 *Assume that:*

(5.1) E *is a nonempty open subset of* $\mathbb{R}^2$,

(5.2) $F : E \longrightarrow \mathbb{R}$ *is a non-negative* $\mathcal{F}$*-function,*

(5.3) $u_* : \mathbb{R} \longrightarrow \mathbb{R}$ *is a minimal solution of* $u' = -F(t,u)$, $u(t_0) = w_0 \geq 0$,

(5.4) $y : [t_0, t_0 + a] \longrightarrow \mathbb{R}$ *is a* C^1 *function such that* $| y(t_0) | \geq w_0$, $(t, | y(t) |) \in E$ *and* $a > 0$,

(5.5) $| y' | \leq F(t, | y |)$ *on* $[t_0, t_0 + a]$.

Then

(5.6) $| y | \geq u_*$

on any common interval of existence of u_* *and* y.

Proof. Since from (5.5)

$$D_- \mid y \mid \geq -F(t, \mid y \mid)$$

on $[t_0, t_0 + a]$ is followed where $D_- \mid y \mid = \lim_{t \longrightarrow t_0^-} \inf \frac{|y(t)| - |t(t_0)|}{t - t_0}$, then (5.4) implies that the function $| y |$ fulfils the assumptions of Theorem 4.2. Thus $| y | \geq u_*$ is implied by this theorem.

Theorem 8 *Let* $f : A \longrightarrow \mathbb{R}^n$ *be a function with* $\mathcal{F}$ *property, where* $A \subseteq \mathbb{R}^{n+1}$ *is a parallelopiped defined by:* $t_0 \leq t \leq t_0 + a$, $| y - y_0 | \leq b$. *Let* $\omega : A_0 \longrightarrow \mathbb{R}$ *be a function with* $\mathcal{F}$ *property, where* A_0 *is defined by* $t_0 < t \leq t_0 + a$, $0 \leq u \leq 2b$. *In addition, assume that* $\omega(t, 0) = 0$ *and the only solution* u *of the differential equation*

(5.7) $$u' = \omega(t, u)$$

on any interval $[t_0, t_0 + \varepsilon]$ *satisfying*

(5.8) $$\lim_{t \longrightarrow t_0^+} u(t) = 0 \quad and \quad \lim_{t \longrightarrow t_0^+} \frac{u(t)}{t - t_0} = 0$$

is $u = 0$ *on* $[t_0, t_0 + a]$. *For* $(t, y_1), (t, y_2) \in A$ *with* $t > t_0$, *let*

(5.9) $\quad | f(t,y_1) - f(t,y_2) | \leq \omega(t, | y_1 - y_2 |).$

Then the initial value problem

(5.10) $\quad y' = f(t,y),\ y(t_0) = y_0$

has at most one solution on any interval $[t_0, t_0+\varepsilon]$.

Proof. Since the condition

(5.11) $\quad \omega(t,0) = 0 \ \text{ for } \ t_0 \leq t \leq t_0 + a$

holds, then, of course, $u = 0$ on $[t_0, t_0+a]$ is a solution of (5.7). Suppose that, for some $\varepsilon > 0$, the initial value problem (5.10) has two distinct solutions y_1 and y_2 on $[t_0, t_0+\varepsilon]$.

Put $y = y_1 - y_2$. Since ε is a positive number, then it can be supposed to be such that $y(t_0+\varepsilon) \neq 0$.

$$| y(t_0+\varepsilon) | = | y_1(t_0+\varepsilon) - y_2(t_0+\varepsilon) | \leq$$

$$| y_1(t_0+\varepsilon) - y_0 | + | y_1(t_0+\varepsilon) - y_0 | \leq 2b,$$

because of $| y - y_0 | < b$. Then $y(t_0) = 0$ and

$$y'(t_0) = y_1'(t_0) - y_1'(t_0) = f(t, y_1(t_0)) - f(t, y_2(t_0)) = f(t,0) - f(t,0) = 0.$$

By (5.9),

$$| y'(t) | = | y_1'(t) - y_1'(t) | = | f(t, y_1(t)) - f(t, y_2(t)) | \leq$$

$$\leq \omega(t, | y_1(t) - y_2(t) |) = \omega(t, | y(t) |)$$

on $(t_0, t_0+\varepsilon]$. Let u_* be a minimal solution of the initial value problem

$$u' = -\omega(t,u), \quad u(t_0+\varepsilon) = | y(t_0+\varepsilon) |,$$

where $| y(t_0+\varepsilon) | \leq 2b$.

Then by Lemma 5.1 we have

(5.12) $\quad | y | \geq u_*$

on any subinterval of $[t_0, t_0+\varepsilon]$, which is a common interval of existence of u_*, y.

By the Extension Theorem 6 in [1] and Lemma 3.1, u_* can be extended as the minimal solution, to the left until $(t, u_0(t))$ approaches to a point of ∂A_0 for some $t \in [t_0, t_0+a]$. During the extension (5.12) holds, so that $(t, u_0))$ comes arbitrarily close to some point $(\delta, 0) \in \partial A_0$ for certain $t_0 < t \leq t_0 + a$, where $\delta \geq t_0$. If $\delta > t_0$, then (5.11) shows that u_0 has an extension on $(t_0, t_0+\varepsilon]$ with $u_0(t) = 0$ for $t \in (t_0, \delta]$.

Thus the maximal interval of existence of u_* is $(t_0, t_0+\varepsilon]$. It follows from (5.12) that $\lim_{t\longrightarrow t_0^+} u_0(t) = 0$. Since

$$\lim_{t\longrightarrow t_0^+} \frac{u_0(t)}{t-t_0} = \lim_{t\longrightarrow t_0^+} \frac{u_0'(t)}{1} = \lim_{t\longrightarrow t_0^+} \omega(t, u_0(t)),$$

then (by (5.11)), $\lim_{t\longrightarrow t_0^+} \frac{u_0(t)}{t-t_0} = 0$. By the assumption concerning to (5.7), $u_* = 0$. Since it contradicts to $u(t_0+\varepsilon) =\mid y(t_0+\varepsilon) \mid\neq 0$, the theorem follows.

Corollary 2 *If $g : [t_0, t_0+a] \longrightarrow \mathbb{R}$ is a bounded derivative, on the interval $[t_0, t_0+a]$, $h : [0, 2b] \longrightarrow \mathbb{R}$ is a bounded derivative, a continuous function on $[0, 2b]$, then $\omega(t, u) = g(t)h(u)$ is admissible in Theorem 5.1 (i.e. the conclusion of Theorem 5.1 holds if (5.3) is replaced by*

$$\mid f(t, y_1) - f(t, y_2) \mid \leq g(t)h(\mid y_1 - y_2 \mid) \tag{5.13}$$

for (t, y_1), $(t, y_2) \in A$ with $t > t_0$.

Proof. First, to verify that ω has $\mathcal{F}$ property, i.e. ω satisfies conditions (i) and (ii).

Ad. (i). Let $t_1 \in (t_0, t_0+a)$ and ε be an arbitrary positive number.

Since h is continuous, then it is locally uniformly continuous. Thus for every ε_1 there exists $\delta_1 > 0$ such that for every u_1, $u_2 \in [0, 2b]$ if $\mid u_1 - u_2 \mid < \delta_1$, then $\mid h(u_1) - h(u_2) \mid < \varepsilon_1$. Since g is bounded, say $\mid g \mid < M$, where M is a positive number, then

$$\mid \omega(t_1, u_1) - \omega(t_1, u_2) \mid = \mid g(t_1)h(u_1) - g(t_1)h(u_2) \mid =$$

$$= \mid g(t_1) \mid \cdot \mid h(u_1) - h(u_{12}) \mid < M \mid h(u_1) - h(u_2) \mid .$$

Thus for every $\varepsilon > 0$ there exists $\delta = \min\{\delta_1, \varepsilon_1/M\}$ such that for every $t_1 \in (t_0, t_0 + a)$ and $u_1, u_2 \in [0, 2b]$

$$\text{if } \mid u_1 - u_2 \mid < \delta \quad \text{then } \mid \omega(t_1, u_1) - \omega(t_1, u_2) \mid < \varepsilon.$$

So we showed that a family $\{\omega_t\}_t$ is equicontinuous.

Ad. (ii). Since for each $u_1 \in [0, 2b]$ the function $g(t)h(u_1)$ is a bounded derivative then ω satisfies condition (ii).

Let $u'(t) = g(t)h(u(t))$. Then

$$u(t) - u(t_0) = \int_{t_0}^{t} g(z)h(u(z))dz.$$

Since $\lim_{t \longrightarrow t_0} u(t) = 0$ then $u(t_0) = 0$.

Note that $u = 0$ on $[t_0, t_0 + a]$ is a solution of (5.7). Since h is a bounded derivative then by the generalization of Picard theorem in [1] it follows that there exists at most one solution on $[t_0, t_0 + a]$. Since $u = 0$ is the only solution of (5.7) on $[t_0, t_0 + a]$, then $w(t, u) = g(t)h(u(t))$ is admissible in Theorem 5.1.

Corollary 3 *Let $F : A \longrightarrow \mathbb{R}$ be a function with $\mathcal{F}$ property, where A is defined by $t_0 \leq t \leq t_0 + a$ and $\mid u - u_0 \mid \leq b$.*
In addition, let F be nonincreasing with respect to u (for fixed t). Then the differential equation $u' = F(t, u)$, $u(t_0) = w_0$ has at most one solution on any interval $[t_0, t_0 + \varepsilon]$, where ε is an arbitrary positive number.

Proof. Since F is nonincreasing then

$$(F(t, u_2) - F(t, u_1)) \cdot (u_2 - u_1) \leq 0,$$

so by Theorem 5.1 there exists at most one solution of

$$u' = F(t, u), u(t_0) = w_0$$

on $[t_0, t_0 + \varepsilon]$, where $\varepsilon > 0$.

6. Uniqueness Theorem

In the following uniqueness theorem conditions are imposed on a family of solutions rather than on f in the differential equation

(6.1) $y' = f(t, y), \quad y(t_0) = y_0.$

A function $f : E \longrightarrow Y$, where $\emptyset \neq E \subset X$, $(X, |\ |_X)$ and $(Y, |\ |_Y)$ are normed spaces, is said to be uniformly Lipschitz continuous on E with respect to $y \in Y$ if there exists a constant $K > 0$ satisfying the condition

$$| f(t, y_1) - f(t, y_2) |_Y \leq K \mid y_1 - y_2 |_X \quad \text{for all } (t, y_i) \in E$$

$$\text{with } i = 1, 2.$$

Theorem 9 *Assume that*

(6.2) $f : A \longrightarrow \mathbb{R}^n$ *is a function with* $\mathcal{F}$ *property, where* A *is defined by* $t_0 \leq t \leq t_0 + a$, $\mid y - y_0 \mid \leq b$,

(6.3) *there exists a function* $\eta(t, t_1, y_1)$ *on* $t_0 \leq t$, $t_1 \leq t_0 + a$, $\mid y_1 - y_0 \mid \leq \beta < b$ *such that*

(6.3.1) $y = \eta(t, t_1, y_1)$ *is a solution of* $y' = f(t, y), y(t_1) = y_1$ *for the point* $(t_1, y_1) \in A$,

(6.3.2) $\eta(t, t_1, y_1)$ *is uniformly Lipschitz continuous with respect to* y_1,

(6.3.3) *if two solutions* $y = \eta(t, t_1, y_2)$, $y = \eta(t, t_2, y_2)$ *pass through the same point* (t, y), *then* $\eta(t, t_1, y_1) = \eta(t, t_2, y_2)$ *for* $t_0 \leq t \leq t_0 + a$.

Then $y = \eta(t, t_0, y_0)$ *is the only solution of (6.1) for* $t_0 \leq t \leq t_0 + a$, $\mid y_1 - y_0 \mid \leq \beta$.

Proof. Let y be any solution of the initial value problem (6.1). We shall show that $y(t) = \eta(t, t_0, y_0)$ for small $\rho = t - t_0 \geq 0$.(6.3.2) implies that

(6.4) $\mid \eta(t, t_1, y_1) - \eta(t, t_1, y_2) \mid \leq K \mid y_1 - y_2 \mid$

for $t_0 \leq t$, $t_1 \leq t_0 + a$ and $\mid y_1 - y_0 \mid \leq \beta$, $\mid y_2 - y_0 \mid \leq \beta$.

From (i) and (ii) it follows that there is $M > 0$ such that $\mid f \mid \leq M$ on A. Then any solution y of (6.1) satisfies

$$\mid y(t) - y_0 \mid \leq M(t - t_0) \leq \frac{1}{2}\beta \quad \text{if} \quad t_0 \leq t \leq t_0 + \frac{\beta}{2M}.$$

Thus $\eta(t,s,y(s))$ is well defined and

$$| \eta(t,s,y(s)) - y(s) | \leq M | t-s | \leq \frac{1}{2}\beta \quad \text{if } t_0 \leq t,\ s \leq t_0 + \frac{\beta}{2M}.$$

Then $| \eta(t,s,y(s)) - y_0 | \leq | \eta(t,s,y(s)) - y(s) | + | y(s) - y_0 | \leq \beta$ for $t_0 \leq t$, $s \leq t_0 + \gamma$, where $\gamma = \min\{a, \frac{\beta}{2M}\}$. Since $y = \eta(t,t_1,y_2)$ and $y_2 = \eta(t_1,s,y(s))$ pass through the point (t_1,y_2), so it follows from (6.3.3) that

$$\eta(t_1,s,y(s)) = \eta(t,t_1,y_2).$$

Put $y_1 = y(t_1)$ and $y_2 = \eta(t_1,s,y(s))$. Then

(6.5) $| \eta(t,t_1,y(t_1)) - \eta(t,s,y(s)) | \leq K | y(t_1) - \eta(t_1,s,y(s)) |$

if $t_0 \leq t$, $s \leq s+\gamma$. Let t be a fixed point from $[t_0, t_0+\gamma]$. It will be verified that

(6.6) $\tau(t) = \eta(t,t_0,y_0) - y(t) = 0.$

Put

(6.7) $\sigma(t) = \eta(t,t_0,y_0) - \eta(t,s,y(s))$ for $t_0 \leq s \leq t$ $(t \leq t_0 + \gamma)$.

Then $\sigma(t_0) = 0$ and $\sigma(t) = \tau(t)$. It follows from (6.5) and (6.7) that

(6.8) $| \sigma(t_1) - \sigma(s) | \leq K | y(t_1) - \eta(t_1,s,y(s)) |$.

Because $y = \eta(t,s,y(s))$ is a solution of the differential equation $y' = f(t,y)$, it passes through the point $(s,y(s))$, so

$$\eta(t_1,s,y(s)) = y(s) + (t_1 - s)f(s,y(s)) + o(1) \quad \text{as } t_1 \longrightarrow s.$$

Also $y(t_1) = y(s) + (t_1 - s)f(s,y(s)) + o(1)$ as t_1 tends to s. (6.8) implies that

$$\sigma(t_1) - \sigma(s) = K \cdot o(1) \,|\, t_1 - s \,| \quad \text{as } t_1 \longrightarrow s;$$

i.e. $\frac{d\sigma}{ds}$ exists and equals to 0. Thus $\sigma(s)$ is constant, $\sigma(t_0) = 0$ and so $\sigma(s) = 0$ for $t_0 \leq s \leq t$. In particular $\tau(t) = \sigma(t)$ satisfies (6.6), and the proof is completed.

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