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# ON UNIQUENESS OF MINIMAL PROJECTION IN $l_1$

Blatter and Cheney [1] studied the existence of a minimal projection[1] from the Banach space $l_1$ of all absolutely summable real sequences onto its condimension one subspace $D = f^{-1}(0)$, where $f \in (l_1)^*$. In particular, they proved that if the linear functional $f$ is of the form:

$$(1) \qquad f = (f_n), \quad 1 = f_1 \geq f_2 \geq \ldots \geq 0, \quad f_3 > 0,$$

Then the hyperplane $f^{-1}(0)$ in the space $l_1$ admits at least one minimal projection, whose norm is greater than 1.

In the present paper we establish a necessary and sufficient condition for the uniquencess of a minimal projection from $l_1$ onto $f^{-1}(0)$ under the assumption (1) . Our terminology and notation follows [1] and [2] .

Let $B$ be a Banach space over the field $R$ of real numbers, $S_B = \{x \in B : \|x\| = 1\}$ , the unit sphere. By $B^*$ we denote the dual of $B$.

For every $f \in B^* \setminus \{0\}$ we shall consider the family of linear operators

$$(2) \qquad P_{f,z} = I - f \otimes z : B \longrightarrow B, \quad z \in B,$$

i.e. $\qquad P_{f,z}(x) = x - f(x)z.$

Let $z \in f^{-1}(1)$, then $P_{f,z}$ is a projection from $B$ onto $D = f^{-1}(0)$, (cf. [1]).

Let

$$(3) \qquad q(f) = \inf \{\|P_{f,z}\| : z \in f^{-1}(1)\},$$

the relative projection constant of the pair $(B, f^{-1}(0))$, and let

(4) $G_f = \{z \in f^{-1}(1) : \|P_{f,z}\| = q(f)\}$,

the set of the points $z$ corresponding to minimal projections. The equality card$(G_f) = 1$ expresses the uniqueness of a minimal projection on $f^{-1}(0)$. For every $z = (z_i) \in l_1$ we write $z > 0$ (resp. $z \geqslant 0$) iff $z_i > 0$ (resp. $z_i \geqslant 0$) for all indices i. We let

$$B_+ = \{z \in l_1 : z \geqslant 0\} .$$

We shall need the following results of he papers [1] and [2] :

Proposition 1.([1], Lemma 4) Let $B = l_1$. If $f \in S_{B^*}$ satisfies the condition (1), then

(5) $G_f = \{z \in l_1 : z \geqslant 0, \|P_{f,z}\| = q(f)\}$.

Proposition 2.([1], Lemma 3). Let $B = l_1$, $f \in S_{B^*}$, and $z \in f^{-1}(1)$.

Then

(6) $\|P_{f,z}\| = \sup_n \{|1 - f_n z_n| + |f_n|(\|z\| - |z_n|)\}$.

In particular, if $f \geqslant 0$ and $z \geqslant 0$, then

(7) $\|P_{f,z}\| = 1 + \sup_i r_i$, where $r_i = (\|z\| - 2z_i)f_i$.

Proposition 3.([1], Theorem 7). Let $B = l_1$ and let $f \in S_{B^*}$ satisfying (1). Assume $1/0 = \infty$, $0 \cdot \infty = 0$ and denote

(8) $a_j = \sum_{i=1}^{j} f_i$, $b_j = \sum_{i=1}^{j} f_i^{-1}$, $\beta_j = b_j/(j-2)$ for $j > 2$ ;

(9) $c_j = \min\{f_j b_{j-1}, a_{j-1}\}$ for $j \geqslant 2$;

(10) $k = k(f) = \max\{j : c_j - j \geqslant -3\}$.

Then

(i) if $f = f^o = (1, 1, \ldots, 1, \ldots)$, then $q(f) = 2$ ;

(ii) if $f \neq f^o$, then $q(f) = 1 + u$, where

$$(11)\quad u = \begin{cases} 2[(\beta_k - 1/f_k)(k-2) + a_k/f_k - k]^{-1}, & \text{if } a_k < k-2, \\ 2(a_k\beta_k - k)^{-1}, & \text{if } a_k \geqslant k-2 \end{cases}$$

and

$$(12)\quad \beta_k \geqslant 1/f_k .$$

Remark 1. By [2], Lemma 3.1, $q(f) < 2$ for $f \neq f^o$. Adapting Proposition 3 to the case of the n-dimensional space $B = l_1^n$ $(n \geqslant 4)$ we get: if $1 = f_1 \geqslant \ldots \geqslant f_n \geqslant 0$, $f_3 > 0$, then $q(f) = 1 + u$ with $u$ defined by (11), (cf. [2] and [3]).

Proposition 4. ([2], Theorem 3.2). Let $B = l_1^n$, $n \geqslant 3$, $f \in S_{B^*}$ satisfying

$$(13)\quad 1 = f_1 \geqslant f_2 \ldots \geqslant f_n \geqslant 0, \quad f_3 > 0.$$

Let

$$(14)\quad m = m(f) = \begin{cases} k(f), & \text{if } 1/k(f) \neq \beta_{k(f)}, \\ \min\{i \in \{3,4,\ldots,k(f)\}: 1/f_{i+1} = \beta_{k(f)}\}, & \text{otherwise.} \end{cases}$$

Then the minimal projection of $B$ onto $f^{-1}(0)$ is unique iff one of the following two conditions is satisfied:

(i) $a_m > m-2$ ;

(ii) $a_m < m-2$, $f_2 < 1$, $a_{m-1} > m-3$ .

Proposition 5. ([2], Theorem 3.3). Let $B = l_1^n$, $n \geqslant 4$, $f \in S_{B^*}$ satisfying (13), $m = m(f) \geqslant 3$, $\tilde{B} = l_1^m$, $\tilde{f} = (f_1, \ldots, f_m) \in \tilde{B}^*$. Then $m(\tilde{f}) = m$, $q(\tilde{f}) = q(f)$ and

$$G_f = \{z \in B_+ : (z_1, \ldots, z_m) \in G_{\tilde{f}},\ z_j = 0 \text{ for } m < j \leq n\};$$

in particular, card $(G_f) =$ card $(G_{\tilde{f}})$.

The next theorem is our main result:

Theorem 1. Let $B = l_1$, $f \in S_{B^*}$ such that $1 = f_1 \geqslant f_2 \geqslant \ldots \geqslant 0$, $f_3 > 0$.

Then

(i) if $f = f^o = (1, 1, \ldots, 1, \ldots)$ then card $(G_f) > 1$;

(ii) if $f \neq f^o$, $\tilde{B} = l_1^m$ and $\tilde{f} = (f_1, \ldots, f_m)$, where m is given by (14), then the minimal projection of B onto $f^{-1}(0)$ is unique iff the minimal projection of $\tilde{B}$ onto $\tilde{f}^{-1}(0)$ is unique.

Proof. (i) Let $f = f^o$ and $z^1 = (1/2, 1/4, 1/4, 0,0,\dots) \in B$, $z^2 = (1/3, 1/3, 1/3, 0,0,\dots) \in B$. Evidently $z^1, z^2 \in f^{-1}(1)$, $P_{f^o,z^1} \neq P_{f^o,z^2}$. By (7), Proposition 3, and Remark 1, $\|P_{f^o,z^1}\| = \|P_{f^o,z^2}\| = q(f^o) = 2$. Hence $\mathrm{card}(G_{f^o}) > 1$.

(ii) Let $f \neq f^o$. By Remark 1, $q(f) = 1 + u < 2$. Let $m = m(f)$ be defined by (14). Let $\tilde{B} = l_1^m$, $\tilde{f} = (f_1, \dots, f_m)$. Recall, that by Proposition 5, $q(\tilde{f}) = q(f)$.

First, we prove that

$$(15)\qquad G_f \subset \{z \in B_+ : (z_1,\dots,z_m) \in G_{\tilde{f}},\ z_i = 0 \text{ for all } i > m\}.$$

To this end let $z \in G_f$ and $\tilde{z} = (z_1,\dots,z_m) \in \tilde{B}$. By (5), $z \in B_+$.

In the case $f_{m+1} = 0$ (and hence $f_j = 0$ for all $j \geq m+1$) we have $\tilde{z} \in \tilde{f}^{-1}(1)$, and, by Proposition 2,

$$q(\tilde{f}) \leq \|P_{\tilde{f},\tilde{z}}\| = 1 + \max\{(\|\tilde{z}\| - 2z_i) f_i : 1 \leq i \leq m\} \leq$$

$$\leq 1 + \max\{(\|z\| - 2z_i) f_i : 1 \leq i \leq m\} = q(f).$$

Therefore $\|z\| = \|\tilde{z}\|$ and hence $z_i = 0$ for all $i \geq m+1$, and $\tilde{z} \in G_{\tilde{f}}$.

Now consider the case: $f_{m+1} \neq 0$. Observe that

$$(16)\qquad z_1 > 0.$$

Indeed, $\|z\| = \|z\|\,\|f\| \geq f(z) = 1$ and by Proposition 2 and Remark 1, $\|z\| - 2z_1 = (\|z\| - 2z_1) f_1 \leq \|P_{f,z}\| - 1 = u < 1$. By (16) and the fact that $z \in B_+$, we have $t = f_1 z_1 + f_2 z_2 + \dots + f_m z_m > 0$. Evidently $\tilde{y} = t^{-1}\tilde{z} \in \tilde{f}^{-1}(1)$.

By Proposition 2 there exists an index $i_o$ such that $1 \leq i_o \leq m$ and $\|P_{\tilde{f},\tilde{y}}\| = 1 + (\|\tilde{y}\| - 2y_{i_o}) f_{i_o}$. Hence

$$tu \leq t(\|P_{\tilde{f},\tilde{y}}\| - 1) = t(\|\tilde{y}\| - 2y_{i_o}) f_{i_o} = (\|t\tilde{y}\| - 2z_{i_o}) f_{i_o} =$$

$$= (\|z\| - \sum_{j=m+1}^{\infty} z_j - 2z_{i_o}) f_{i_o} = f_{i_o}(\|z\| - 2z_{i_o}) - f_{i_o} \sum_{j=m+1}^{\infty} z_j$$

$$\leq u - f_{i_o} \sum_{j=m+1}^{\infty} z_j ,$$

i.e.

$$f_{i_0}\sum_{j=m+1}^{\infty} z_j \le u(1 - t) = u \sum_{j=m+1}^{\infty} f_j z_j .$$

Since $f_j \le f_{i_0}$ for all $j \geqslant m+1$ and $u < 1$ (by Remark 1), we get $z_j = 0$ for all $j \geqslant m+1$. Moreover, $\|P_{\tilde f,\tilde z}\| = \|P_{f,z}\| = q(f) = g(\tilde f)$, $\tilde z \in G_{\tilde f}$. This establishes (15).

Finally we prove that

(17) $\quad G_f \supset \{z \in B_+ : (z_1,\dots,z_m) \in G_{\tilde f},\ z_i = 0 \text{ for all } i \geqslant m+1\}.$

To this end let $\tilde z \in G_{\tilde f}$, $\tilde z = (z_1,\dots,z_m)$, and $z = (z_1,\dots,z_m,0,\dots) \in B$. By Proposition 5 $\tilde z \in \tilde B_+$ and therefore $z \in B_+$. By Proposition 2 and by (1)

$$\|P_{f,z}\| = 1 + \max\{u,\ f_{m+1}\|z\|\},$$

In the case: $f_{m+1} = 0$, by Propositions 3 and 5, we have

$$\|P_{f,z}\| = \|P_{\tilde f,\tilde z}\| = q(\tilde f) = q(f) \text{ and } z \in G_f .$$

In the case where $f_{m+1} > 0$, we let $\tilde{\tilde f} = (f_1,\dots,f_{m+1})$, $\tilde{\tilde B} = l_1^{m+1}$, $\tilde{\tilde z} = (z_1,\dots,z_m,0) \in \tilde{\tilde B}$. By Proposition 5,

$$G_{\tilde{\tilde f}} = \{x \in B_+ : (x_1,\dots,x_m) \in G_{\tilde f},\ x_{m+1} = 0\}$$

and $\|P_{\tilde{\tilde f},\tilde{\tilde z}}\| = \|P_{\tilde f,\tilde z}\| = q(\tilde f) = q(\tilde{\tilde f}) = 1 + u$. Hence $f_{m+1}\|\tilde{\tilde z}\| \le u$, because $\|P_{\tilde{\tilde f},\tilde{\tilde z}}\| = 1 + \max\{u,\ f_{m+1}\|\tilde{\tilde z}\|\} = 1 + u$. Observe next that $\|\tilde z\| = \|\tilde{\tilde z}\| = \|z\|$. Therefore

$$\|P_{f,z}\| = 1 + \max\{u, f_{m+1}\|z\|\} = 1 + u = q(f) \text{ and } z \in G_f.$$

<u>Example 1</u>. Let $f = (2^{1-n}) \in (l_1)^*$. By (10) and (14), $m(f) = k(f) = 3$. Hence, by Theorem 1 and Proposition 4, the minimal projection from $l_1$ onto $f^{-1}(0)$ is unique.

<u>Remark 2</u>. Since the space $l_1$ is symmetric, our Theorem 1 yields a necessary and sufficien condition for the uniqueness of minimal projection from $l_1$ onto its subspace $g^{-1}(0)$ in the case that there is a permutation $(p(n))$ of positive integers and the sequence $(\varepsilon_n)$ $\varepsilon_i = +1$ or $-1$ such that the linear functional $f = (\varepsilon_n g_{p(n)}) \in (l_1)^*$ satisfies the condition (1). As far as I know, the remaining case is unexplored.

Note

1) The projection (= an idempotent bounded linear operator) $P_o$ from a Banach space B onto its subspace D is minimal if $\|P_o\| \leq \|P\|$ for every projection $P : B \rightarrow D$.

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Streszczenie

W artykule podano dowód konieczności i dostateczności warunków, dla których rzut z minimalną normą $> 1$, odwzorowujący przestrzeń $l_1$ na podprzestrzeń Ker f, gdzie
$f = (f_1, f_2, \ldots) \in (l_1)^*$ i
$1 = f_1 \geq f_2 \ldots \geq 0$; $f_3 > 0$, jest jedyny.
Podane warunki dla funkcjonału f są obecnie prawdopodobnie jedynymi znanymi warunkami, przy których wiadomo, że istnieje rzut z minimalną normą $> 1$ z $l_1$ na Ker f.