ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
Problemy Matematyczne 1985 z.7

WŁODZIMIERZ ŚLĘZAK
WSP w Bydgoszczy

# ON ABSOLUTE EXTENSORS

## 1. Introduction and preliminaries

A topological space $Y$ is called an absolute extensor for metric spaces, briefly AE, if , whenever $X$ is a metric space and $A$ is a closed subset of $X$ , then any continuous function from A into Y can be extended to a continuous function from the whole space $X$ into $Y$ (cf. [15],[2], pp.66-70). It is well-known that a convex subset of a locally convex linear topological space is an AE (see [3] and [8]). In [11], pp. 187-189 J. Dugundji proves that also all real vector spaces with the finite (Whitehead) topology (which need not be locally convex nor even linear topological spaces) have the property of being an AE. In [7], theorem 2.4 , this property is proved for spaces with so-called local convex structure (see [16], cf. also [10] and [12]). Generalizing the concept of local equiconnectedness introduced by Fox in [12] and studied in [7],[10],[16],[9], Lech Pasicki recently developed the theory of S- contractibility and defined two classes of spaces connected with this notion (see [18]-[21]). The purpose of this note is to investigate the possibility of replacing the class of locally convex linear topological spaces in Dugundji's extension formula by a more general class of S- contractible spaces. We repeat the notions related to S- contractibility.

A set $Y$ is S-linear if there is a mapping $S : Y \times [0;1] \times Y \to Y$ such that $S(a,0,b)= b$ and $S(a,1,b)= a$ for all $a,b \in Y$. Note that the pair $(Y,S)$ is a convex prestructure in the sense of [13] .

For any subset $B$ of S-linear set $Y$ define

$$(1)\quad \operatorname{co} S(B):= \bigcap \{D \subset Y : B \subset S^*(B \times [0;1] \times D) \subset D\}$$

where $S^*(B \times [0;1] \times D) := \bigcup \{S(a,t,b) : a \in B,\ b \in D,\ 0 \leq t \leq 1\}$.
For $B = \emptyset$ let $\operatorname{co} S(B) := \emptyset$ .

It is easily checked, that $B \subset \operatorname{co} S(B)$ so that above definition is correct.

A topological space Y is locally S- contractible if Y is S-linear and for every $y \in Y$ there exists a neighbourhood U of y such that for every $a \in U$, the value of an exponential mapping $G([S(a,\cdot,\cdot)]) = \hat{g}_a : [0,1] \to C(U, Y)$ defined by formulas

(2) $\hat{g}_a(t) := f_t$, $U \ni b \mapsto f_t(b) = S_1(a,t,b) \in Y$ ,

$S_1 = S|U \times [0;1] \times U$ denoting restriction of S, is continuous, where the set $C(U,Y)$ of all continuous maps from U to Y is equipped with the quasi-compact-open topology. If $U = Y$ for all $y \in Y$, then Y is called S-contractible.

By using the properties of quasi-compact-open topology (contained in many handbooks on homotopy theory, e.g. Sze-Tsen Hu) we may formulate an equivalent definition (cf. [19], p.596):

An S- linear topological space Y is S- contractible if $S(a,\cdot,\cdot) : [0,1] \times Y \to Y$ is a homotopy joining the identity $S(a,0,\cdot) = id_Y$ with a constant map $S(a,1,\cdot) = const_a$.

A topological space Y is locally of Pasicki type I if it is locally S-contractible for an S satisfying the following condition:

(3) For every $y \in Y$ and any neighbourhood V of y there exists a neighbourhood U of y such that $\operatorname{co} S\, U \subset V$.

If Y is S- contractible and the above condition (3) holds, then Y is called to be of Pasicki type I. In the special case when S is continuous on $U \times [0,1] \times U$ as a function of 3 variables the above definition agrees with [7] , p. $103_{4-3}$ .

We introduce some more general notion:

DEFINITION: An S-contractible space Y is of type m if, for every metric space X and every continuous map $f : X \to Y$ the following is true;

(4) For each $x \in X$ and each neighbourhood $V \subset Y$ of $f(x)$

there is a neighbourhood $W_x \subset X$ of $x \in X$ and some subset $C \subset Y$ such that

$$f * W_x \subset C \subset \text{co}\, S \quad C \subset V .$$

REMARK 1. This class of S-contractible spaces includes all Pasicki type I spaces. Indeed, put $C = U \subset \text{co}S(U)$ in (4) in order to obtain (3) . Notice, that in our definition the set C may fails to be open. On the other hand, the class of S-contractible spaces of type m includes all Dugundji affine spaces of type m ( see [11], p.187), in particular all real vector spaces with the finite topology for an S defined as follows :

(5) $S(a,t,b) = ta + (1-t)\cdot b$ .

This also shows that our definition is essentially more general, since each vector space Y being S-contractible for an S defined by (5) is locally convex, while vector spaces with the finite topology, as is shown in the appendix to [11], may fail to be linear topological spaces.

2. A generalization of Dugundji extension theorem

We are now in a position to state and prove our main theorem:

THEOREM. Let (X,d) be an arbitrary metric space, $A = \text{cl}\, A \subset X$ a closed subset, and Y an S-contractible space of type m. Then each continuous map $f : A \to Y$ has a continuous extension $E(f): X \to Y$ such that

(6) $E(f) * X \subset [\text{co}S(f * A)]$

PROOF: For each $x \in X-A$ let $B_x$ be an open ball centered at x with radius $r(x) < \text{dist}(x,A)/2$ . The family $\{B_x ; x \in X-A\}$ is an open cover of the paracompact X-A, so it has a neighbourhood-finite refinement $\{U_t : t \in T\}$ . Let $B(A, 2r) := \{x_1 \in X : \text{dist}(x_1, A) < 2r\}$. Observe that a ball $B_x$ centered outside B(A ,2r) cannot intersect B(A,r). Indeed, pick $x_1 \in B_x$ and observe that

$\text{dist}(x_1, A) \geq \text{dist}(x,A) - d(x_1,x) > \text{dist}(x,A) - \text{dist}(x,A)/2 =$
$= \text{dist}(x,A)/2 > r$ . Consequently any $U_t$ that intersects B(A,r) is contained in a $B_x$ centered within B(A, 2r) and so

has diameter $\operatorname{diam} U_t := \sup\{d(x_1,x_2);\ (x_1,x_2)\in U_t \times U_t\} \le$ $\le \operatorname{diam} B_x \le 2\,r(x) \le \operatorname{dist}(x,A) \le 2r$ .
With each (nonempty) $U_t$ associate a point $a_t \in A$ as follows: choose an $x_t \in U_t$ and find $a_t \in A$ with $d(x_t, a_t) < 2 \operatorname{dist}(x_t, A)$. The fundamental property of "Dugundji system" $\{(U_t, a_t) : t\in T\}$ is:
(7) for each a A and each neighbourhood
$W_a$ in X, there is a beighbourhood $V_a \subset W_a$ such that

$$U_t \cap V_a \neq \emptyset \text{ implies } [U_t \subset W_a] \text{ and } a_t \in A \cap W_a .$$

Indeed, we can assume $W_a = B(a,r)$. Taking $V_a = B(a, r/12)$, any $U_t$ intersecting $V_a$ has diameter $\operatorname{diam} U_t \le r/6$ so that it is completely within $B(a, r/4)$. For any such $t$ we have $d(x_t,a) < r/4$ , so that $\operatorname{dist}(x_t, A) < r/4$ and also

$$d(a_t,a) \le d(a_t,x_t) + d(x_t,a) \le 2d(x_t, a_t) + \operatorname{dist}(x_t,A) < 3r/4 ;$$

that is $a_t \in W_a$. Suppose that $\prec$ is a total ordering of the set T. Let $\{k_t : t\in T\}$ be a partition of unity on X-A subordinated to $\{U_t : t\in T\}$ . For each $x \in X-A$ define

$$T_x := \{t \in T : k_t(x) \neq 0\} =: \{t_1, t_2, \ldots, t_n\}$$

where $n = n(x)$ and $t_1 \prec t_2 \prec \ldots \prec t_n$. Put
(8) $c_t(x) := k_t(x)/ \max\{k_t(x) : t \in T_x\}$
and define $E(f) : X \to Y$ by formula (cf. [1]):

$$(9)\quad E(f)(x) := \begin{cases} f(x); & x\in A \\ S(b_{t_1}, c_{t_1}(x), S(b_{t_2}, c_{t_2}(x), S(\ldots \\ \ldots, S(b_{t_n}, c_{t_n}(x), b), \ldots) ; & x\in X-A \end{cases}$$

where $b_t := f(a_t)$ and $b=f(a)$ for an arbitrary but fixed element a of A. It is easily seen that there always exists $t\in T_x$ such that $c_t(x)= 1$ ; then $S(f(a_t), c_t(x), f(a)) = f(a_t)$ for $a\in A$, thus our definition of E(f) is correct. An idea of replacing the convex combination by the iterations of S is due by L. Pasicki in context of fixed points theory ([20], p. 173). It is necessary to stress that in formula (9) we take $t_1 \in T_x$ .
But for each $x_o \in X\setminus A$ there is a neighbourhood $O(x_o)$ of $x_o$ such that the following equivalence holds:

$$c_t(x) = 0 \iff t\in T\setminus T_{x_o} \quad \text{whenever} \quad x\in O(x_o).$$

Consequently for all $x \in O(x_o)$ we essentially take in (9) those

$b_{t_i}$, for which $t_i \in T_{x_o}$. Observe, that the function $O(x_o) \ni x \to g_n(x) := S(b_{t_n}, c_{t_n}(x), b)$ is continuous on $O(x_o)$. For $i = 1,2,\dots,n-1$ let us define recursively

(10) $O(x_o) \ni x \to g_{n-i}(x) := S(b_{t_{n-i}}, c_{t_{n-i}}(x), g_{n-i+1}(x)) \in Y.$

Since $b_{t_{n-i}}$ are constant on $O(x_o)$ and $S(b_{t_{n-i}}, \cdot, \cdot)$: $[0,1] \times Y \to Y$ is jointly continuous as a homotopy, we deduce that each $g_{n-i}$ is continuous on $O(x_o)$ being a superposition of continuous maps. Thus $E(f)|O(x_o) = g_1$ is continuous on $O(x_o)$. Since $\{O(x_o) : x_o \in X \setminus A\}$ forms an open cover of $X \setminus A$, we deduce that $E(f)$ is continuous at each point of the open set $X \setminus A$.

The complete the proof we shall show the continuity of $E(f)$ at each point of A. Pick $a \in A$ and let $W \subset Y$ be an arbitrary neighbourhood of $f(a) = E(f)(a)$. Since $Y$ is of type m and $f$ is continuous, there is a neighbourhood $W_a$ in $X$ such that

$$f * [W_a \cap A] \subset C \subset coS\ C \subset W$$

for some subset $C$ in $Y$. Find $V_a \subset W_a$ satisfying the condition in (7) : we will show $E(f) * V_a \subset W$. Since each $U_t$, $t \in T_x$ intersects $V_a$, the corresponding $a_t$, $t \in T_x$ all lie in $A \cap W_a$, so that the $f(a_t)$ are all elements of $C$. According to (1) and (9) we find $E(f)(x) \in coS(C)$. Thus $E(f) * V_a \subset W$ and $E(f)$ is continuous at a . Since $E(f)$ is continuous at each point of $X$, the map $E(f) : X \to Y$ is continuous.

The formula (9) shows that $E(f)$ is an extension of $f$.

To show that $E(f) * X \subset coS(f * A)$ choose any subset $D$ belonging to the family under the sign of intersection in formula (1), where $B = f * A$ . If $x \in X \setminus A$ then, in accordance with (9),

$g_n(x) := S(b_{t_n}, c_{t_n}(x), b) \in S * (B \times [0,1] \times D) \subset D$

because of $b = f(a) \in B \subset D$ and $b_{t_n} = f(a_{t_n}) \in B$, $c_{t_n}(x) \in [0,1]$ .

Observe that for $i = 1,2,\dots,n-1$ we have recursively

$g_{n-i}(x) \in S * (B \times [0,1] \times D) \subset D$, for a $g_{n-i}$ defined by (10).

This yields $g_1(x) = E(f)(x) \in coS(B)$. Since $D$ was arbitrary, we obtain $E(f)(x) = coS(B)$.

If $x \in A$ then $E(f)(x) = f(x) \in B \subset coS(B)$. Thus in both cases we have $E(f) * X \subset coS(f * A)$

REMARK O. Observe that the points $a_t$ and functions $k_t$ are independent on the funotion f, so that the operator E: $C(A, Y) \rightarrow C(X, Y)$ is universal in some sense.

COROLLARY: If f is a Borel measurable function of the first Borel class from a Borel subset X of a complete metric space to a metrizable S-contractible type m space Y, then f is the pointwise limit of a sequence of continuous functions from X to Y .

3. Pathological examples of S-contractibility

Let us recall, that a oonvex prestruoture is a nonempty set Y together with a map from $Y \times [0;1] \times Y$ to Y. We think of S as a set of elements that can be blended or mixted, and S(a,t,b) denotes a blend of a and b in which the concentration (or portion) of a is t and the concentration of b is 1-t . A convex structure is a convex prestructure (Y,S) satisfying the following five postulates :

P.1. $S(a,t,b)= S(b,1-t,a)$ for all $t \in [0;1]$ ; $a,b \in Y$

P.2. $S(a,t, S(b,u,c)) = S(S(a,t\cdot[t + (1-t)u]^{-1},b),$ $t + (1-t)\cdot u, c)$ for all $t,u \in [0;1]$ with $t + (1-t)\cdot u \neq 0$ and a,b,c belonging to Y

P.3. $S(a,t,a) = a$ for all $t \in [0;1]$ , $a \in Y$ .

P.4. If $S(a,t,b) = S(a,t,c)$ for some $t \neq 1$ and some $a \in Y$, then $b = c$;

P.5. $S(a,0,b)= b$ for all $a,b \in Y$.

In the early 1940's J. von Neuman and O. Morgenstein employed abstract convex structures in their theory of games and economic behavior. Important contributions were made a few years later by M. Stone [24] . He called such a structure a barycentric calculus. Since then, convex structures have been applied to studies in color vision, utility theory, quantum mechanics and petroleum engineering (see [13] for more informations and further references).

Note that none of the postulates P.1-P.4 is in general not fulfilled by S-linear (Y,S) , cf. Ex. 2 below, in which P.1-P.2. and P.4. are violated and [13] , ex 3, p.986 for the failure of p. P.3.

Note, that coS : $2^Y \rightarrow 2^Y$ is not an hull-operator in any reasonable sense. In [17] a map $co : 2^Y \to 2^Y$ is said to be a convex hull-operator on Y, if, denoting by $\hat{F}(Y)$ the familly of all finite subsets of Y , the following postulates are all satisfied :

H.1. $co\ \emptyset = \emptyset$

H.2. $co(\{x\}) = \{x\}$ for all $x \in X$

H.3. $co\ (co\ A) = co\ A$ for $A \in 2^Y$

H.4. $co\ A = \bigcup\{co\ F : F \subset A,\ F \in \hat{F}(Y)\}$ .

It is easy to check, that any operator satisfying H.1.-H.4. is monotone, that co A is the smallest convex set containing A, that the intersection of an arbitrary family of convex sets is also a convex set (A is called convex if A = co A), etc. For our co S there exist some example showing that in general $coS \circ coS \neq coS$ (cf. Ex. 2 below), and that S-hull of finite subset may fails to be compact (see undermentioned Ex. 3).

Example 1 (privately communicated by Dr Lech Pasicki)

Let $Y := \{z \in C : re\ z \geq -1\}$ be the complex halfplane with induced topology. Consider a half-moon $A \subset Y$ defined by $A := \{z \in Y : |z| < 2$ and $|z+1| \geq sqrt\ 5\}$ . For $a,b \in Y$ define $L(a,b)$ as the symmetry axis of the segment $\overline{ab}$ , viz.

$$L(a,b) := \{z \in C : re[(b-a)\cdot(z - 2^{-1}(a+b))^*] = 0\},$$

where $z^*$ is the complex conjugate of z. Define also $L_0 := \{z \in C : re\ z = -2\}$ . Then consider a mapping $S : Y \times [0;1] \times Y \to Y$ defined by formula

$$S(a,t,b) := \begin{cases} z_0 + |a-z_0|\cdot\exp[i(t\arg a + (1-t)\cdot\arg b)] \\ \text{iff } L(a,b) \cap L_0 = \{z_0\} \text{ and } re\ a > 0 \\ ta + (1-t)b \text{ if } im\ a = im\ b \text{, so that } L(a,b) \cap L_0 = \emptyset \\ \text{or if } -1 \leq re\ a \leq 0 . \end{cases}$$

It is easy to observe that our half-moon A is S-convex, viz. $coS\ A = A$, and that, denoting by cl the usual closure operator on Y, $cl\ coS\ A = cl\ A$ while

$$coS\ (cl\ A) = \{z \in Y ; |z| \geq 2 \text{ and } re\ z \geq 0\}$$

is essentialy larger than $cl(coS\ A)$ .

EXAMPLE 2. Let Y denote the real line with usual topology .

For $a,b \in Y$ and for $t \in [0;1]$ define

$$S(a,t,b) := 2(b-a)t^2 + 3(a-b)\cdot t + b .$$

Observe that S fulfils only P.3. and P.5. Taking $A = \{-1,1\}$ it is easy to verify, that $coS \circ coS \neq coS$ in contrast to H.3. .

EXAMPLE 3. Let Y be as in Ex.2 and let

$$S(a,t,b) := \begin{cases} 3(a-b)\cdot t + b & \text{for } 0 \leq t \leq 2/3 \\ -3(a+b)\, t + a - 3(a+b) & \text{for } 2/3 \leq t \leq 1 \end{cases}$$

Observe that $coS(\{-1,1\}) = Y$ and thus is noncompact.

The reader is reffered to [6], [22], [5], [7], [10], [16] for further interesting and important examples and to [14], [4], [23], [17], [13] for information about others existing kinds of generalized convexity.

I wish to express my thanks to the referees, Prof. M. Kisielewicz and Dr L. Rybiński for theirs precious remarks allowing to eliminate some incorrectness in the first draft of this paper.

REFERENCES

[1] Arens R., Extensions of functions on fully normal spaces. Pacific J. Math., 2 (1952) pp. 11-22

[2] Bessaga Cz., Pełczyński A. , Selected topics in infinite-dimesnional topology, Monografie Matematyczne 58, PWN, W-wa 1975

[3] Borsuk K., Über Isomorphie der Functionalräume, Bull. Int. Acad. Polon. Sci., Ser. A (1933) pp. 1-10

[4] Bryant V.W., Webster R.J., Convexity spaces I, J. Math. Anal. Appl., 37 (1972) pp. 206-213

[5] Burago J.D., Załgaller W.A., Zbiory wypukłe w przestrzeniach Riemanna o nieujemnej krzywiźnie w jęz. ros. , UMN, 32, no3 (1977), PP.3-55

[6] Susemann H., The geometry of geodesics, Academic Press, New York 1965.

[7] Curtis D.W., Some theorems and examples on local equi-connectedness and its specializations. Fundamenta Math., 72 (1971) pp.101-113

[8] Dugundji J., An extension of Tietze's theorem, Pacific J. Math., 1 (1951) pp.353-367

[9] Dugundji J., Absolute neighbourhood retracts and local connectedness in arbitrary metric spaces, Comp. Math., 13 (1958) pp.229-246

[10] Dugundji J., Locally equiconnected spaces and absolute neighbourhood retracts, Fundamenta Math., 57 (1965) pp.187-193

[11] Dugundji J., Topology, Allyn and Bacon, Boston, Mass. 1970

[12] Fox R.M., On fiber spaces II, Bull. Amer. Math. Soc., 49 1943 pp.733-735

[13] Gudder S., Chroeck F., Generalized convexity, SIAM J. Math. Anal. XI, 6 (1980) pp.984-1001

[14] Hammer P.C., Semispaces and the topology of convexity. In "Convexity" , Amer. Math. Soc. Proceedings of Symposia in Pure Mathematics, VII, Convexity (1963) pp.305-316

[15] Hanner O., Retraction and extension of maps of metric and non-metric spaces. Ark. Math. 2 (1952) pp.315-359

[16] Himmelberg C.J., Some theorems on equiconnected and locally equiconnected spaces Trans.Amer. Soc. 115 (1965) pp.43-53

[17] Komiya H., Convexity on a topological space, Fundamenta Math. CXI.2. (1981) 107-113

[18] Pasicki L., On the Cellina theorem of non-empty intersection, Rev. Roum. Math. Pures et Appl. , XXV, 7 (1980) pp.1095-1097

[19] Pasicki L., Retracts in metric spaces, Proc. Amer. Math. Soc. LXXVIII, 4 (1980) pp.595-600

[20] Pasicki L., Three fixed point theorems, Bull. Acad. Polon. des Sci. XXVIII, 3-4 (1980) pp.173-175

[21] Pasicki L., A fixed point theory for multi-valued mappings, Proc. Amer. Math. Soc.LXXXIII, 4 (1981) pp.781-789

[22] Prenowitz W., Geometry of Join Systems, New York, Reinhard and Winston, 1969

[23] de Smet H.J.P., Algebraic Convesity, Bull. Soc. Math. Belg. Sér. B, 31 (1979), no2 pp.173-182

[24] Stone M., Postulates for a barycentric calculus, Ann. of Math. 29 (1949) pp.25-30

ABSTRACT

L. Pasicki has introduced S-contractible spaces of type I as a generalization of locally convex linear topological spaces. In this paper we introduce a larger type of S-contractible spaces and we prove an analogue of Dugundji extension formula for continuous functions with values in S-contractible space of this new type m. Also some connections between S-contractible spaces and so-called preconvex structures are explained.

O ABSOLUTNYCH EXTENSORACH

Streszczenie

W pracy uogólnia się znane twierdzenie Dugundjiego o przedłużaniu funkcji ciągłych o wartościach w lokalnie wypukłej przestrzeni liniowo-topologicznej na przypadek funkcji o wartościach we wprowadzonych przez Pasickiego przestrzeniach S-ściągalnych z abstrakcyjnie określoną strukturą wypukłą. Dyskutuje się też związki tych przestrzeni z różnymi rodzajami uogólnionej wypukłości.