ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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SOME MULTIPLICATIVE PROPERTIES OF SETS OF REAL NUMBERS

Through this paper, the real line will be denoted by R and m denotes Lebesgue measure in R.
If $A,B \subseteq R$, then $A+B = \{a+b : a \in A, b \in B\}$, $A \cdot B = \{a \cdot b : a \in A, b \in B\}$,
$\frac{A}{B} = \left\{\frac{a}{b} : a \in A , b \in B - \{0\}\right\}$, $\exp A = \{\exp a : a \in A\}$ ,
$\ln A = \{\ln a : a > 0, a \in A\}$ and $\sqrt{A} = \{\sqrt{a} : a \geq 0 , a \in A\}$ .

P. Erdös, K. Kunen and R.D. Mauldin proved in [2] the following theorems. 1) If P is a nonempty perfect subset of R, then there is a perfect set M with Lebesgue measure zero so that $P + M = R$ .

2)(CH) There is a subset X of R such that $|X| = \mathfrak{c}$ and if $m(G)= 0$ then $m(G + X)= 0$ for each $G \subseteq X$ .

In [4] Steinhaus proved that if A and B are linear sets of positive Lebesgue measure, then the set A + B contains an interval (cf.[1])
The same conclusion holds when A and B have the property of Baire and are of the second category (Piccard [3] , cf[1]).

J. Ceder and D.K. Ganguly proved in [1] that there exists a second category set A such that A + A does not contain an interval.

In this note we investigate the multiplicative analogues of these results .

REMARK 1. a) If $A \subseteq R$ is of the first category, then exp A is of the first category .

b) If $A \subseteq (0,\infty)$ is of the first category, then ln A is of the first category .

Remark 1 follows because the functions exp and ln are homeomorphism of the spaces R and $(0,\infty)$.

Remark 2. a) If $A \subseteq R$ and $m(A)= 0$ then $m(\exp A) = 0$ .

b) If $A \subseteq (0,\infty)$ and $m(A) = 0$ then $m(\ln A) = 0$.

Proof. a) Let $A_n = A \cap (-\infty, n)$. Since $\exp A = \bigcup_{n \in N} \exp A_n$, it suffices to show $m(\exp A_n) = 0$ for $n = 1, 2, \dots$.
Notice that if $G$ is an open set in $(-\infty, n)$, then $m(\exp G) \leq \leq (\exp n) \cdot m(G)$. If $m(A_n) = 0$, then for any $\varepsilon > 0$ there is an open set $G \subseteq (-\infty, n)$ such that $m(G) \leq \exp(-n)\varepsilon$ and $A_n \subseteq G$. Then for any $\varepsilon > 0$ there is an open set $\exp G$ such that $\exp A_n \subseteq \exp G$ and $m(\exp G) \leq \varepsilon$. Hence $m(\exp A_n) = 0$

The proof of b) is similar.

PROPOSITION 1. Let $P$ be a nonempty perfect subset of $R$. Then there is a perfect set $M$ so that $m(M) = 0$ and $P \cdot M = R$.

Since $M$ is a perfect set and $m(M) = 0$, $M$ is of the first category.

Proof. Let $P$ be a nonempty perfect subset of $R$. If there is a nonempty perfect subset $T \subseteq P \cap (0, \infty)$ then $\ln T$ is a perfect subset of $R$. Then there is a perfect set $N$ so that $m(N) = 0$ and $\ln T + N = R$. Then $T = \exp(\ln T)$ and for $M = \exp N \cup \{0\} \cup \cup (-\exp N)$ we have $P \cdot M \supseteq T \cdot M = \exp(\ln T) \cdot \exp N \cup \exp(\ln T) \cdot \cdot (-\exp N) \cup \{0\} = \exp(\ln T + N) \cup \{0\} \cup (-\exp(\ln T + N)) = R$.
Assume that $P \subseteq (-\infty, 0)$. Then there is a nonempty perfect subset $T \subset P \subseteq (-\infty, 0)$. Then there is a perfect set $N$ so that $m(N) = 0$ and $\ln(-T) + N = R$. Then for $M = \exp N \cup \{0\} \cup \cup (-\exp N)$ we have $-P \cdot M \supseteq -T \cdot M = R$. Since $M = -M$, $-P \cdot M = = (-P) \cdot (-M) = P \cdot M$.

PROPOSITION 2. (CH) There is a subset $X$ of $R$ such that $|X| = \mathfrak{c}$ and if $m(I) = 0$ then $m(X \cdot I) = 0$ for each $I \subseteq R$.

Proof. Let $B$ a subset of $R$ such that $|B| = \mathfrak{c}$ and if $m(I) = 0$ then $m(B + I) = 0$ for every $I \subseteq R$ and $X = \exp B$.
Let $I = J \cup K \cup L$, where $J = I \cap (-\infty, 0)$, $K = I \cap \{0\}$ and $L = I \cap (0, \infty)$. Since $X \cdot I = X \cdot J \cup X \cdot K \cup X \cdot L$, it suffices to show $m(X \cdot J) = = m(X \cdot K) =$

$= m(X \cdot L) = 0$. Since $m(\ln J) = 0$, $m(B + \ln J) = 0$ and $m(X \cdot J) = m(\exp(B + \ln J)) = 0$.

It is clear that $m(X \cdot K) = 0$.

Since $m(\ln(-L)) = 0$, $m(B + \ln(-L)) = 0$ and $m(X \cdot L) = = m(-(X \cdot (-L))) = m(X \cdot (-L)) = 0$.

PROPOSITION 3. a) If $A$ and $B$ are sets of positive Lebesgue measure, then the set $A \cdot B$ contains a nonempty interval.
b) If $A$ and $B$ are sets of the second category and $A,B$ have the property of Baire, then the set $A \cdot B$ contains a nonempty interval.
Proposition 3 follows from theorems of Steinhaus [4] and Piccard [3]. The proof of this proposition is similar to the proof of Proposition 2.
PROPOSITION 4. Let $C$ be the set of cardinality less than continuum.

a) Then there is a set $A$ of the second category such that $C \cap A \cdot A = \emptyset$.

b) Then there is a set $A$ of the full external measure such that $C \cap A \cdot A = \emptyset$.

Proof. Let $(G_\xi)_{\xi<\omega}$ be an well-ordering of all residual $G_\delta$ subsets of the line. Choose $a_0 \in G_0 - \{0\}$ and

$$a_\gamma \in G_\gamma - \left(\frac{C}{\{a_\beta : \beta<\gamma\}} \cup \{0\} \cup \pm\sqrt{C}\right).$$

Let $A = \{a_\gamma : \gamma < \omega_c\}$. Then $A$ is of the second category because it intersects each residual $G_\delta$ set.
Suppose that there is $c \in C \cap A \cdot A$. Then $c = a_\beta \cdot a_\gamma$ and $\beta \leq \gamma$.
If $\beta < \gamma$ then $a_\gamma = \frac{c}{a_\beta}$ - a contradition. If $\beta = \gamma$ then $a_\gamma = \pm\sqrt{c}$ and it is impossible. Hence $C \cap A \cdot A = \emptyset$.
The proof of b) is similar.

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## O PEWNYCH MULTYPLIKATYWNYCH WŁASNOŚCIACH ZBIORÓW LICZB RZECZYWISTYCH


### Streszczenie

W pracy odnotowano następujące fakty:

1. Jeśli $P \subseteq R$ jest niepustym zbiorem doskonałym, to istnieje zbiór doskonały M taki, że miara Lebesgue'a zbioru M jest równa zero oraz $P \cdot M = R$, gdzie $P \cdot M$ oznacza zbiór $\{xy : x \in P, y \in M\}$.
2. (CH) Istnieje zbiór $X \subseteq R$ mocy continuum taki, że jeśli J jest zbiorem miary Lebesgue'a zero, to zbiór $X \cdot J$ jest miary zero,
3. Jeżeli $A, B \subseteq R$ są zbiorami o dodatniej mierze Lebesgue'a (drugiej kategorii i z własnością Baire'a), to zbiór $A \cdot B$ zawiera odcinek niepusty,
4. Dla dowolnego zbioru C mocy mniejszej niż continuum istnieje zbiór drugiej kategorii (miary zewnętrznej pełnej) taki, że $C \cap A \cdot A = \emptyset$.