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STANISŁAW SZYMAŃSKI
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A SYSTEM OF AXIOMS OF THE METRICAL GEOMETRY ON SPHERE

Let S_o be a given sphere in three dimensional Euclidean space E_3. We shall denote by J_o the set of all isometries of the space E_3 that preserve this sphere S_o. By the geometry on sphere we understand the theory investigating the properties of objects which are composed by using the points of the set S_o and which are invariant with respect to the group J_o (see [7], p. 130). It doesn't allow mistake the geometry on the sphere with the elliptical geometry. The model of this last one is the sphere S_o on which the antipodal points are identified (see [6], p. 49). The great circles on sphere S_o play a similar role as the lines in the Euclidean geometry on the plane. However, in this case the properties of triangles are described by the spherical trigonometry. Therefore the reader shall find many theorems concerning the geometry on the sphere in the handbook on the spherical trigonometry as for example [7],[8].

In this paper the axiomatical conception of the geometry on the sphere is presented, where the primitive notions are the set of all points belonging to the sphere and the function which is called the distance of points, which assignes to every couple of points a non-negative real number, which is equal to the angular measure of a central angle of the sphere, whose arms pass by these points. A choice of primary notions accepted in this paper was suggested by papers [2] and [3] concerning Euclidean and hyperbolic geometry on the plane. Also the given axioms were formulated on pattern of the axioms appearing in the mentioned papers. With regard to the specifity of the geometry on the sphere, the manner of development of the theory supported on given system of axioms differs from manners adapted in papers [2] and [3] . The formal aspect of all demonstrations is analogous as in the script [3] . The results obtained from the given axiomatic are divided into two parts, the first of which establish the substance of this paper. There is demonstrated, that this axiomatic is satisfied by some model of the metric geometry on the sphere, called the basic model $\not{S}_o$. The known theorems of spherical trigonometry and of vector calculus there were used (see resp. [7], [8] and [4]). In the second part, whioh will be the subject of the next articles, the theory supported on our axiomatic will be developed in such extend, that as an effect will be obtained the proof, that every model of this theory is isomorphic with the basic model S_o. For defining the geometry on sphere we will need the well known model of the spherical trigonometry.

Let S_o be the fixed sphere in threedimensional Euclidean space E_3 possesing the radius r_o. The line of intersection of the sphere S_o with the plane is a circle, which radius r is not greater than r_o. In case, when the plane is passing by the conter of the sphere, the equality $r = r_o$ arise and the obtained circle is named the great circle. In another case where the inequality $r < r_o$ holds, the obtained circle is called the small one.

Two arbitrary points P_1 and P_2 on the sphere which are not antipodal - that means they do not be places on this same diameter of the sphere - determine unambiguously the great circle going by both points. The points P_1 and P_2 are the ends of two arcs of this circle, one of which is greater than the half-circle and the other one is smaller. That smaller arc determine a geodesic line of the sphere passing by the points P_1 and P_2. This means, it is the nearest connexion of the points P_1 and P_2 by the line placed on the sphere. Its length we named a spherical length d_o of these points. The spherical length of two antipodal points is equal to the length of the great halfcircle and has the value $r_o \pi$. For the comfort without loss of generality we admit the radius of the sphere S_o as a unity of arcs measure .

The sphere S_o together with the funotion d_o from the Cartesian product $S_o \times S_o$ into the set of nonnegative real numbers determines the model $\mathfrak{S}_o = \langle S_o, d_o \rangle$ of the metrical geometry on the sphere. This model may be also investigated by using an analytical apparatus. We suppose that in the space E_3 a fixed, right-twisting ortonormal system of reference is given, with basic points $0, E_1, E_2, E_3$. In the presence of this system, for every point of the unit sphere, which has the center at a point 0, there correspond three real numbers x,y,z satisfying the condition $x^2 + y^2 + z^2 = 1$. The situation of the point on the sphere in this method is determined by Cartesian ortogonal coordinates respected to the basic planes of the given system of reference. Denoting by $\vec{p}_1(x_1,y_1,z_1)$; $\vec{p}_2(x_2,y_2,z_2)$ the position-vectors of points P_1 and P_2 respectively, the spherical length of those points will be expressed by formula $d_o(P_1,P_2) = \arccos(\vec{p}_1 \circ \vec{p}_2)$ The set of all position vectors of the points situated on the sphere S_o, or the set of all unit vectors, we shall denote without any risk of misunderstanding also by the sign S_o. The geometry on the sphere may be defined as the categroic (see [5] , p.305) theory having all models isomorphio to $\mathfrak{S}_o$. The aim of the presented paper is to give the proof, that the following axioms

A.0. $\bigvee_{A,B} \; 0 \neq AB \neq \pi$

A.1. $[a \geq 0 \wedge b \geq 0 \wedge (|a-b| \leq aB \leq a+b) \wedge a+b + AB \leq 2\pi] \Longleftrightarrow$
$\Longleftrightarrow \bigvee_{C} (CA = a \wedge BC = b)$

A.2. $[a > 0 \wedge b > 0 \wedge (|a-b| < AB < a+b) \wedge a+b + AB < 2\pi \Longrightarrow$
$\Longrightarrow \bigvee_{2C} (AC = a \wedge BC = b)$

A.3. $[A \neq B \wedge AB + BC = AC \wedge \cos AB \cos BD = \cos AD] \Longrightarrow$
$\Longrightarrow \cos CB \cos BD = \cos CD$

may be accepted as a full axiomatic of the geometry on the sphere. In those axioms the capital letters denote allways the points while the small letters denote the real numbers, and the symbol $d(A,B)$ is replaced by AB.

We read the sign $\bigvee_{2C}$ in A.2 "there exist exactly two points C such that ... ". To be precise, we want to prove the following theorem :

THEOREM I . Let S be the set of elements called a points. Let d be a function defined on the product S x S, which assignes to each pair of points A, B a nonnegative real number called the length between these points and designed by the symbol AB.

The n the necessary and sufficient condition for that a pair $\langle S, d\rangle$ would be isomorphic with the basic model $\#_o = \langle S_o, d_o \rangle$ of the geometry on the sphere is, that S and d are in compliance with axioms A.0 - A.3.

This theorem I in a natural way falls to pieces on the two theorems:

THEOREM I A. Axioms A.0 - A.3 are satisfied in the basic model.

THEOREM I B. Each model of axioms A.0 - A.3 is isomorphic with $\#_o$. The proof of the first one of these theorems is relatively short and determines the object of the present article.

Just the opposite, the proof of the theorem I B requires the enlargement of the complicated theory supported on the system of axioms A.0 - A.3 and will be the subject of our considerations in the next articles.

The geometry on the sphere is a domain for which an earlier theory is the arithmetic of real numbers together with the trigonometry. In particular we will be interested in some function of three variable defined with help of the following definition:

$$Q(x,y,z) := \sin p \cdot \sin(p-x)\sin(p-y)\sin(p-z) \qquad (L1)$$

where p designes a half of the sum of the numbers x,y,z, viz. $2p = x+y+z$. The function $Q(x,y,z)$ has a precise connection with the spherical excess and the measure of a triangle on the sphere. In fact, if we denote by x,y,z the lengths of the three sides of the triangle on our sphere, the spherical excess of this traingle will be expressed by the formula :

$$\sin \frac{\varepsilon}{2} = \frac{\sqrt{Q(x,y,z)}}{2 \cos(x/2) \cos(y/2) \cos(z/2)}$$

and the measure of a triangle on the sphere with the unit radius equals to ε

Using the trigonometrical transformations to the formula (LI) we obtain

$$Q(x,y,z) = 4^{-1} \begin{vmatrix} 1 & \cos x & \cos y \\ \cos x & 1 & \cos z \\ \cos y & \cos z & 1 \end{vmatrix} \qquad (L2)$$

Proof. $(L.1) \Rightarrow Q(x,y,z) = 4^{-1} \left(4 \sin \frac{x,y,z}{2} \sin \frac{x+y-z}{2} \cdot \right.$
$\sin \frac{x-y+z}{2} \sin \frac{-x+y+z}{2} = -\frac{1}{4} \left[\cos x - \cos(y-x)\right]\left[\cos x - \cos(y+z)\right]$
$= -\frac{1}{4}(\cos x - \cos y \cos z - \sin y \sin z)\cdot(\cos x - \cos y \cos z +$
$+ \sin y \sin z) = -\frac{1}{4}\left[(\cos x - \cos y \cos z)^2 - \sin^2 y \sin^2 z\right] =$
$= -\frac{1}{4}\left(\cos^2 x - 2 \cos x \cos y \cos z + \cos^2 y \cos^2 z - \sin^2 y\right.$
$\left.\sin^2 z\right) = \frac{1}{4}(1 + 2 \cos x \cos y \cos z - \cos^2 x - \cos^2 y - \cos^2 z).$

By virtue of the symmetry of the formula (L1) i.e. the fact, that all three variables x,y,z play the same role, it is possible to change their succesion in an arbitrary means. Therefore the following theorem arrives:

$$Q(x,y,z) = Q(x,z,y) = Q(y,x,z) = Q(y,z,x) = Q(z,x,y) = Q(z,y,x) \qquad (L3)$$

By substituting for x,y,z different special values, we obtain from the formula (L2) the equations

$$Q(x,y,0) = 4^{-1} (\cos x - \cos y)^2 \qquad (L4)$$
$$Q(x,y,x+y) = 0 \qquad (L5)$$
$$Q(x,y,\pi) = 4^{-1} (\cos x + \cos y)^2 \qquad (L6)$$

and others special cases of (L2).

We will restrict our further investigations to the properties of the function $Q(x,y,z)$ for the arguments from the interval $[0,\pi]$ only. If all four factors in the formula (L1) are nonnegative, the function $Q(x,y,z)$ will be nonnegative too.

This property of Q(x,y,z) is expressed by the lemma:

$$\bigwedge_{x,y,z\in[0,\pi]} \{[x+y \geq z \wedge x+z \geq y \wedge y+z \geq x \wedge x+y+z \leq 2\pi] \Rightarrow Q(x,y,z) \geq 0\} \quad (L7)$$

Now, we shall prove the next lemma.

$$\bigwedge_{x,y,z\in[0,\pi]} [Q(x,y,z) \geq 0 \Longrightarrow x+y \geq z] \quad (L8)$$

Proof. Let us suppose that our lemma is false. Hence, for certain numbers x,y,z the following relations arrive:

$$x,y,z \in [0,\pi] \quad (1)$$
$$Q(x,y,z) \geq 0 \quad (2)$$
$$x+y < z \quad (3)$$

From assumption (1) and (3) it follows, that

$$z > 0 \quad (4)$$

and therefore in connection with (1) and (3) we obtain

$$0 < x+y+z < 2\pi \quad (5)$$

From (1) and (3) it follows at once, that

$$-2\pi < x+y-z < 0 \quad (6)$$

The inequalities (2) ,(5) and (6) entail in the presence of (L1) the inequality:

$$\sin\frac{x-y+z}{2} \quad \sin\frac{y+z-x}{2} \leq 0 \quad (7).$$

We will show that this inequality is impossible. In fact, if (7) were true, then either

$$\sin\frac{x-y+z}{2} \leq 0 \quad \text{or} \quad \sin\frac{y+z-x}{2} \leq 0 .$$

Each of these two inequalities leads to the contradiction. Indeed, if we suppose that

$$\sin\frac{x-y+z}{2} \leq 0 \quad (8),$$

then because of the assumption (1) we must obtain

$$-2\pi < x-y+z \leq 0 \quad (9) .$$

Adding by sides the inequalities (9) and (6) we obtain

$-4\pi < 2x < 0$, or $-2\pi < x < 0$ what is in contradiction with assumption (1) .

In the either case, when

$$\sin \frac{-x+y+z}{2} \leq 0$$

we reach at a contradiction with assumption (1), analogously as in the case of assumption (8) . This contradiction ends the proof of (L8).

Let us remark, that the formula (L1) is valid for an arbitrary permutation of the arguments x,y,z (cf. (L3)) and from this reason we may draw at once from L8 the conclusion that the relation

$$\bigwedge_{x,y,z \in [0,\pi]} \left[Q(x,y,z) \geq 0 \implies (x+y \geq z \wedge x+z \geq y \wedge y+z \geq x) \right] \quad (L9)$$

holds. In a similar way we are able to prove (L10):

$$\bigwedge_{x,y,z \in [0,\pi]} \left[Q(x,y,z) \geq 0 \implies x+y+z \leq 2\pi \right] \quad (L10)$$

Proof. Let us suppose, that our lemma (L10) is false, so that for some values x,y,z the following relations

$$y,x,z \in [0,\pi] \quad (1)$$
$$Q(x,y,z) \geq 0 \quad (2)$$
$$x+y+z \geq 2\pi \quad (3)$$

are simultaneously satisfied. From the assumptions (1) and (3) the following inequalities

$$2\pi < x+y+z \leq 3\pi \quad (4)$$
$$x,y,z > 0 \quad (5)$$ result .

The inequalities (2) and (4) entail, because of (L1), the inequality

$$\sin \frac{x+y-z}{2} \sin \frac{-x+y+z}{2} \sin \frac{x-y+z}{2} \leq 0 \quad (6)$$

It is satisfied if one of the factors or all three factors simultaneously are smaller than zero or equals to zero.

The inequality (6) is not possible, because the assumption that whichever of the factors is smaller or equal to zero, leads always to the contradiction with the assumption (1).

For instance, let us suppose, that, e.g.

$$\sin \frac{x+y-z}{2} \leqslant 0 \qquad (7)$$

From the inequalities (7) and (5) and from the assumption (1) it follows that

$$-2\pi < x+y-z \leq 0 \qquad (8).$$

After multiplication of the both sides of this inequality by -1 and addition of it to the inequality (4) we obtain the inequalities $\pi < z < \frac{5}{2}\pi$, contrary to our assumption.

Lemmas (L7),(L9) and (L10) give a condition equivalent to the assertion, that the function $Q(x,y,z)$ is taking only non-negative values for x,y,z running an interval $[0,\pi]$. All of these three lemmas can be unified as the following single lemma:

$$\bigwedge_{x,y,z\in[0,\pi]} \left[Q(x,y,z) \geqslant 0 \Leftrightarrow (x+y+z \leq 2\pi \wedge x+y \geqslant z \wedge x+z \geqslant y \wedge y+z \geqslant x)\right] \qquad (L11)$$

A similar lemma for sharp inequalities is also valid, namely

$$\bigwedge_{x,y,z\in[0,\pi]} \left[Q(x,y,z) > 0 \Leftrightarrow (x+y+z < 2\pi \wedge x+y > z \wedge x+z > y \wedge y+z > x)\right] \qquad (L12)$$

We shall prove one more lemma (L13):

$$\bigwedge_{x,y,z\in[0,\pi]} \left[Q(x,y,z) = 0 \Leftrightarrow (x+y=z \vee x+z=y \vee y+z=x \vee x+y+z=2\pi)\right] \qquad (L13)$$

Proof. 1) If some one of the equalities:

$x+y=z$ (1)

$x+z=y$ (2)

$y+z=x$ (3)

$x+y+z=2$ (4) occurs, thus in compliance with (L1)

the inequality $Q(x,y,z)=0$ (5)

must be rightful.

2) Conversely, if the equality (5) holds, then at least one of the fours factors on the right side of the equality (L1) of necessity equals zero. If the first factor is equal to zero, then the assumption that $x\in[0,\pi]$ implies the equality (4) or the equality $x+y+z = 0$, from which it follows suc-

cesively (1), (2) and (3) . If the second factor is equal to zero, it will arrive either (3) or the equality $y+z-x=2$, which will implice $x=0$ and, in effect (4) , The case, where the third or the fourth factor is equal to zero may be treated in an analogous way. It is easy to see that the lemma(L12)may be obtained as an immediate corollary resulting from lemmas (L11) (L11) and (L13). We are now ready to give the proof of above mentioned Theorem I A. That proof relies heavily on the testing that the four mathematical statements $\sharp_o(A,0)$, $\sharp_o(A.1)$, $\sharp_o(A.2)$, $\sharp_o(A.3)$- confirming that in the model $\sharp_o$ the axioms A.0 ÷ A.3 are adequately valid-create true theorems of the spherical trigonometry. We shall verify them succesively.

$$\sharp_o(A.0)\ \bigvee_{\vec{u}_1,\vec{u}_2 \in S_o} \left[\vec{u}_1 \neq \vec{u}_2 \wedge \text{arc cos}\,(\vec{u}_1 \cdot \vec{u}_2) \neq \pi\right]$$

Proof. For the proof of this step it suffices to remark that, in particular, vectors $\vec{u}_1 = \langle 1,0,0 \rangle$ and $\vec{u}_2 = \langle 0,1,0 \rangle$ belong to the sphere S_o, that these vectors are different and $\text{arc cos}\,(\vec{u}_1 \cdot \vec{u}_2) = \frac{\pi}{2}$.

$$\sharp_o(A.1)\ \bigwedge_{\vec{u}_1,\vec{u}_2 \in S_o} \Big\{\big[(1)\ a \geqslant 0 \wedge (2)\ b \geqslant 0 \wedge (3)(|a-b| \leqslant \text{arc cos}\,(\vec{u}_1 \cdot \vec{u}_2) \leq a+b) \wedge (4)\ a+b+\text{arc cos}\,(\vec{u}_1 \cdot \vec{u}_2) \leq 2\pi\big] \Leftrightarrow \bigvee_{\vec{u} \in S_o} \left[\text{arc cos}\,(\vec{u}_1 \cdot \vec{u}) = a \wedge \text{arc cos}\,(\vec{u}_2 \cdot \vec{u}) = b\right]\Big\}.$$

Proof. 1) In the first part of the proof we shall demonstrate, that the system of equations:

$$\begin{cases} \vec{u}_1 \circ \vec{u} = \cos a \\ \vec{u}_2 \circ \vec{u} = \cos b \end{cases} \qquad (5)$$

posses at least one solution $\vec{u} \in S_o$ provided the inequalities (1),(2),(3) and (4) are fulfiled. From these inequalities it follows

$$a,\ b,\ \text{arc cos}\,(\vec{u}_1 \circ \vec{u}_2) \in [0,\pi] \qquad (6)$$

In turn, from (3),(4) and (6) it follows by virtue of (L11) that

$$Q(a,b,\ \text{arc cos}\ (\vec{u}_1 \cdot \vec{u}_2)) \geq 0 \qquad (7)\ ,\quad \text{or}$$

$$1 - (\vec{u}_1 \cdot \vec{u}_2)^2 \geq \cos^2 a + \cos^2 b - 2(\vec{u}_1 \cdot \vec{u}_2) \cos a \cos b \quad (8)$$

Now, we shall distinguish two cases:

1a) Let us make a supplementary assumption that $\vec{u}_1 \times \vec{u}_2 \neq \vec{0}$. Therefore the system of equations (5) may be viewed as an edge - equation of some straight line. The distance of this line from the origin of the system of coordinates (= from the center of the sphere S_0) is expressed by well-known formula:

$$d = \frac{|(\vec{u}_1 \times \vec{u}_2) \times \vec{u}_3|}{|\vec{u}_1 \times \vec{u}_2|} \qquad (9)$$

in which $\vec{u}_3$ denotes a position-vector of an arbitrary point situated on the line under consideration, in other words an arbitrary vector u which satisfies the system of.equations (5). Transforming the formula (9) we obtain :

$$d^2 = \frac{[\vec{u}_2(\vec{u}_1 \cdot \vec{u}_3) - \vec{u}_1(\vec{u}_2 \cdot \vec{u}_3)]^2}{(\vec{u}_1 \times \vec{u}_2)^2} = \frac{[\vec{u}_2 \cos a - \vec{u}_1 \cos b]^2}{\vec{u}_1^{\,2}\, \vec{u}_2^{\,2} - (\vec{u}_1 \cdot \vec{u}_2)^2} =$$

$$= \frac{\cos^2 a + \cos^2 b - 2\,(\vec{u}_1 \cdot \vec{u}_2) \cos a \cos b}{1 - (\vec{u}_1 \cdot \vec{u}_2)^2}$$

then applying (8) we conclude, that $d^2 \leq 1$.

Hence, the investigated line posses at least one point common with the sphere S_0, so there exists a solution $\vec{u} \in S_0$ of the system of equations (5) .

1b) New we suppose that $\vec{u}_1 \times \vec{u}_2 = \vec{0}$. The assumption $\vec{u}_1, \vec{u}_2 \in S_0$ implies the alternative

$$\vec{u}_1 = \vec{u}_2 \quad (10') \qquad \text{or} \qquad \vec{u}_1 = -\vec{u}_2 \quad (10'')$$

from where we deduce the following alternative:

$$\text{arc cos}\ (\vec{u}_1 \cdot \vec{u}_2) = 0 \quad (11') \qquad \text{or arc cos}\ (\vec{u}_1 \cdot \vec{u}_2) = \pi \quad (11'')$$

Using (7) we conclude by virtue of (L4), resp. by virtue of (L6), that the alternative:

$$\cos a - \cos b = 0 \quad (12') \qquad \text{or} \cos a + \cos b = 0 \quad (12'')$$

is valid. In the both cases under examination, the system of equations (5) take up to single equation: $\vec{u}_1 \cdot \vec{u} = \cos a$, which posses always a solution $\vec{u} \in S_o$.

2) It is well known, that in the model $\$_o$ the spherical distance is nonnegative and that the distances of three arbitrary points belonging to the sphere S_o fulfil the triangle inequality and moreover their sum doesn't exceed the number 2π. Hence it follows at once that the inequalities (1),(2),(3) and (4) witll be satisfied provided the system of equations (5) has a solution $\vec{u} \in S_o$.

$$\$_o \text{ (A.2) } \bigwedge_{\vec{u}_1,\vec{u}_2 \in S_o} \{[(1)\ a > 0 \wedge (2)\ b > 0 \wedge (3)(|a-b| < \arc\cos(\vec{u}_1 \cdot \vec{u}_2) < a+b \wedge (4)\ a+b + \arc\cos(\vec{u}_1 \cdot \vec{u}_2) < 2\pi] \Rightarrow$$
$$\Rightarrow \bigvee_{2\,\vec{u} \in S_o} [\arc\cos(\vec{u}_1 \cdot \vec{u}) = a \wedge \arc\cos(\vec{u}_2 \cdot \vec{u}) = b]\}$$

Proof. It follows from assumptions (1),(2),(3) and (4) that $a \in (0,\pi)$ and $b \in (0,\pi)$ (5) and that

$$\arc\cos(\vec{u}_1 \circ \vec{u}_2) \in (0,\pi) \qquad (6)$$

By virtue of (L12) we obtain an inequality

$$Q(a,b\ ,\ \arc\cos\ (\vec{u}_1 \cdot \vec{u}_2)) > 0\ ,\ \text{or equivalently}$$

$$1 - (\vec{u}_1, \vec{u}_2)^2 > \cos^2 a + \cos^2 b - 2(\vec{u}_1 \circ \vec{u}_2) \cos a \ \cos b \quad (7)$$

From (6) it follows that $\vec{u}_1 \cdot \vec{u}_2 \neq \pm 1$, so that $\vec{u}_1 \times \vec{u}_2 \neq \vec{0}$ and therefore the system of equations

$$\begin{cases} \vec{u}_1 \cdot \vec{u} = \cos a \\ \vec{u}_2 \cdot \vec{u} = \cos b \end{cases} \qquad (8)$$

determines an equation of a straight line. The distance such a line from the center of the sphere S_o is given by formula (9) from the previous proof. It is not difficult to calculate that in the presence of the inequality (7) this distance is smaller than 1. Hence the investigated line is crossing over the sphere exactly at two points. Therefore there exist exactly two vectors $\vec{u} \in S_o$ satisfying the system of equations (8) .

$\mathfrak{H}_o$ A.3 $\bigwedge_{\substack{\vec{u}_i \in S_o \\ i \in \{1,2,3,4\}}} \Big[(1)\ \vec{u}_1 \neq \vec{u}_2 \wedge (2) \arccos(\vec{u}_1 \cdot \vec{u}_2) +$

$+ \arccos(\vec{u}_2 \cdot \vec{u}_3) = \arccos(\vec{u}_1 \cdot \vec{u}_3) \wedge (3)(\vec{u}_1 \cdot \vec{u}_2)(\vec{u}_2 \cdot \vec{u}_4) =$

$= (\vec{u}_1 \cdot \vec{u}_4) \Rightarrow (4)(\vec{u}_2 \cdot \vec{u}_3)(\vec{u}_2 \cdot \vec{u}_4) = \vec{u}_3 \cdot \vec{u}_4 \Big]$.

Proof. We have $\vec{u}_i \in S_o \implies (5)\quad \vec{u}_i^{\,2} = 1$. Introducing a notation $\vec{u}_{ij} := \vec{u}_i \times \vec{u}_j$, by using (5) and the Laplace identity we obtain from the assumption (3) the equality

$$\vec{u}_{12} \cdot \vec{u}_{24} = 0 \qquad (6)$$

Subsequently from the basic law of cosines of the sum of angles and from the assumption (2) we obtain:

$$(\vec{u}_1 \cdot \vec{u}_2)(\vec{u}_2 \cdot \vec{u}_3) - \sqrt{1 - (\vec{u}_1 \cdot \vec{u}_2)^2} \cdot \sqrt{1 - (\vec{u}_2 \cdot \vec{u}_3)^2} = \vec{u}_1 \cdot \vec{u}_3 \quad (7)$$

Transforming the equality (7) and using again the Laplace identity, we obtain

$$\vec{u}_{12}^{\,2} \cdot \vec{u}_{23}^{\,2} - (\vec{u}_{12} \cdot \vec{u}_{23})^2 = 0, \text{ or equivalently}$$

$$(\vec{u}_{12} \times \vec{u}_{23})^2 = 0 \qquad (8).$$

From 1 and (8) the collinearity of vector $\vec{u}_1 \times \vec{u}_2$ and $\vec{u}_2 \times \vec{u}_3$ easily follows. Hence, applying (6) we obtain

$$\vec{u}_{23} \cdot \vec{u}_{24} = 0 \qquad (9).$$

The relation (9) in the presence of (3) by the Laplace identity is equivalent to the thesis of our theorem.

So we see, that the received axiomatic is fulfiled in the basic model $\mathfrak{H}_o$ of the geometry on the sphere. In the subsequent papers the theory based on this system of axioms shall be developed and the proof of its categoric property will appear.

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PEWNA AKSJOMATYKA GEOMETRII METRYCZNEJ NA SFERZE

Streszczenie

W artykule przedstawiono aksjomatykę geometrii na sferze oraz podano dowód, że aksjomatyka ta jest spełniona w pewnym modelu geometrii metrycznej na sferze, nazwanym podstawowym modelem S_0. W dalszych artykułach zostanie rozwinięta teoria oparta na tej aksjomatyce i podany dowód jej kategoryczności.