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ON THE CLASSIFICATION OF SET-VALUED FUNCTIONS

Using a refined Baire classification of multifunctions (i.e. closed set-valued functions), we present here some new results on multifunctions which are similar in spirit to some of the wellknown theorems on functions due to Baire, Lebesgue, Hausdorff, Hahn and Banach. These results were developed in connection with a new notion of derivative [3] which is set-valued.

1. Introduction

Given any topological space (Y, \mathcal{U}) , let $P(Y)$ denote the space of all closed subsets of Y . Define, for each $U \in \mathcal{U}$,

$$U_* = \{F \in P(Y) : F \subset U\} \quad \text{and} \quad U^* = \{F \in P(Y) : F \cap U \neq \emptyset\}.$$

We shall use the Vietoris topology on the space $P(Y)$ which is generated by the family $\{U_* : U \in \mathcal{U}\} \cup \{U^* : U \in \mathcal{U}\}$.

Given any other topological space X , let $M(X, Y)$ denote the space of all multifunctions $\varphi : X \rightarrow P(Y)$. We shall assume throughout that X is a perfect space, viz. every open set in X is an F_σ -set. Let $M^*(X, Y)$ and $M_k(X, Y)$ denote, further, the spaces of multifunctions in $M(X, Y)$ whose values are nonempty or compact respectively. When Y is a topological vector space, we use similarly $M_c(X, Y)$ to denote the space of convex-valued multifunctions in $M(X, Y)$. We set, further, $M_{k,c}(X, Y) = M_k(X, Y) \cap M_c(X, Y)$, $M_k^*(X, Y) = M^*(X, Y) \cap M_k(X, Y)$, etc.

Next, given any countable ordinal number α , a multifunction $\varphi \in M(X, Y)$ will be said to be of Baire class α if for each open set G in $P(Y)$ the set $\varphi^{-1}(G)$ is of additive class α in X . Let φ be said, further, to be of lower or upper Baire class α if for each open set U in Y the

set $\varphi^{-1}(U^*)$ or $\varphi^{-1}(U)$ respectively is of additive class α in X .

We shall use B_α to denote the Baire class α and LB_α and UB_α to denote the lower and upper Baire classes α respectively. Thus LB_0 and UB_0 are the well-known classes of lower and upper semicontinuous multifunctions respectively, or briefly LSC and USC multifunctions respectively, and B_0 is simply the class of continuous multifunctions (see e.g. [5, p. 173]).

The classes LB_α and UB_α have been considered by Kuratowski in one of his recent papers [6], under a different notation, where many of the known results on LSC and USC multifunctions have been extended to the general classes LB_α and UB_α . The results presented here are however new even for $\alpha = 0$.

Given any pair of multifunctions $\varphi, \psi \in M(X, Y)$, we say $\varphi \subset \psi$ provided $\varphi(x) \subset \psi(x)$ for each $x \in X$. The multifunctions $\varphi \cup \psi$ and $\varphi \cap \psi$ are defined in turn by

$$(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x), (\varphi \cap \psi)(x) = \varphi(x) \cap \psi(x), x \in X.$$

Next, a sequence of multifunctions (φ_n) in $M(X, Y)$ will be said to converge to a multifunction $\varphi \in M(X, Y)$ if the sequence $\{\varphi_n(x)\}$ converges to $\varphi(x)$ in the Vietoris topology of $P(Y)$ for each $x \in X$. Further, $\{\varphi_n\}$ will be called non-decreasing or nonincreasing if $\varphi_n \subset \varphi_{n+1}$ or $\varphi_n \supset \varphi_{n+1}$ respectively for each n . The multifunctions $\bigcup_{n=1}^{\infty} \varphi_n$ and $\bigcap_{n=1}^{\infty} \varphi_n$ are defined in turn by

$$\left(\bigcup_{n=1}^{\infty} \varphi_n\right)(x) = \left\{ \bigcup_{n=1}^{\infty} \varphi_n(x) \right\}^-, \left(\bigcap_{n=1}^{\infty} \varphi_n\right)(x) = \bigcap_{n=1}^{\infty} \varphi_n(x), x \in X$$

Finally, if there exists a finite set of functions $f_i: X \rightarrow Y$ ($i=1, 2, \dots, n$) such that $\varphi = \bigcup_{i=1}^n f_i$, then φ will be called an elementary multifunction and $\{f_i: i=1, 2, \dots, n\}$ will be called the elements of φ .

We state here further a few elementary results on multifunctions. These results have been obtained in [6] in the case when Y is a compact metric space. Unless stated otherwise, we shall always assume α to be an arbitrary countable ordinal

number.

It is clear from the definitions that a multifunction B_0 iff $\varphi \in LB_0 \cap UB_0$. For $\alpha > 0$ we have, on the other hand,

1.1 THEOREM. If $\varphi \in M_k(X, Y)$ and Y is second countable, then $\varphi \in B_\alpha$ iff $\varphi \in LB_\alpha \cap UB_\alpha$.

As regards the mutual relationship between the lower and upper Baire classes, we have

1.2 THEOREM. Suppose $\varphi \in M(X, Y)$ where Y is a perfect space.

(a) If $\varphi \in UB_\alpha$, then $\varphi \in LB_{\alpha+1}$.

(b) If φ is compact-valued and in LB_α , and if Y is further normal, then $\varphi \in UB_{\alpha+1}$.

Thus if $\varphi \in M_k(X, Y)$, where Y is a second countable perfect space, and if either (i) UB_α or (ii) Y is normal and $\varphi \in LB_\alpha$, then $\varphi \in B_{\alpha+1}$.

1.3 THEOREM. If two multifunctions $\varphi, \psi \in M(X, Y)$ are both in LB_α or in UB_α , then so is $\varphi \cup \psi$.

Hence if all the elements of some elementary multifunction $\varphi \in M(X, Y)$ are in B_α , it follows from Theorems 1.1 and 1.3 that $\varphi \in B_\alpha$ provided Y is second countable, and this holds for $\alpha = 0$ without any hypothesis on Y .

To avoid further definitions, most of the results are presented here under the hypothesis of separability on Y . In the forthcoming paper we develop, however, a theory which applies to nonseparable Y as well. An application of this theory yields extension of many results on the Baire classification of functions to functions with a nonseparable range. It yields also new results on the lower and upper Baire classes of real-valued functions which are defined similar to the Baire's definitions [1] of lower and upper semicontinuous functions.

2. Characterizations of LB_α and UB_α

We present here characterizations of multifunctions in LB_α and UB_α in terms of limits of monotone sequences of multi-

functions in B_α . These results are similar to the Baire's characterizations of lower and upper semicontinuous real-valued functions (see [1] or [4, p. 28d]). There is however no duality between the classes LB_α and UB_α , and hence their characterizations are independent of each other.

Let us state first a general result on the Baire class of limits of monotone sequences of multifunctions.

2.1. THEOREM. Let (φ_n) be a sequence of multifunctions in $M(X, Y)$ and suppose Y is T_1 .

(a) If $\{\varphi_n\}$ is nondecreasing and $\varphi_n \in LB_\alpha$ for each n , then $\{\varphi_n\}$ converges to $\bigcup_{n=1}^{\infty} \varphi_n$ which is in LB_α .

(b) If $\{\varphi_n\}$ is nonincreasing and φ_n is compact-valued and in UB_α for each n , then $\{\varphi_n\}$ converges to $\bigcap_{n=1}^{\infty} \varphi_n$ which is in UB_α .

2.2. THEOREM. Suppose Y is a separable metric space, $\varphi \in M^*(X, Y)$ is complete-valued and that $\alpha > 0$. Then $\varphi \in LB_\alpha$ iff it is the limit of a nondecreasing sequence of elementary multifunctions $\{\varphi_n\}$ whose elements are in B_α .

The above theorem is easily seen to contain the selection theorem of Kuratowski and Ryll-Nardzewski [7]. It does not hold in general for $\alpha = 0$. But we do have in that case the following theorem on convex-valued multifunctions which in turn contains two selection theorems of Michael [9].

2.3. THEOREM. Suppose Y is a metrizable locally convex space, $\varphi \in M_c^*(X, Y)$ and that one of the following two conditions holds:

(i) X is normal, Y is separable and φ is complete-valued

(ii) X is collectionwise normal and φ is compact valued

Then φ is LSC iff it is the limit of a nondecreasing sequence of elementary multifunctions $\{\varphi_n\}$ whose elements are continuous.

For the multifunctions in UB_α we have similarly the following characterizations which hold for $\alpha > 0$ and $\alpha = 0$.

2.4. THEOREM. Suppose Y is a locally compact separable metric space, $\varphi \in M_k(X, Y)$ and that $\alpha > 0$. Then $\varphi \in UB_\alpha$ iff it is the limit of a nonincreasing sequence of compact-valued

which generalizes the existing results in that direction (see e.g. [5, p. 391] and Banach [2]).

Next, we obtain with the help of the above theorems an extension of the classical Lebesgue-Hausdorff theorem [5, p. 393] to multifunctions. The analytic classes of multifunctions are defined as usual using transfinite induction. The analytic class 0 consists of all continuous multifunctions and, for each countable ordinal $\alpha > 0$, the analytic class α is defined to be the class of all pointwise limits of sequences of multifunctions of analytic classes lower than α .

3.4. THEOREM. Suppose $\varphi \in M_k^*(X, Y)$, where X is normal and Y is a separable metrizable absolute retract. Then φ is in analytic class ∞ iff it is in B_α or $B_{\alpha+1}$ according as α is finite or infinite.

The same holds for a function $f: X \rightarrow Y$ in terms of its analytic class as a function.

The last part of the above theorem generalizes all the existing versions of the Lebesgue-Hausdorff theorem due to Lebesgue [8], Hausdorff [4], Kuratowski [5 p. 393] and Banach [2].

4. Interposition theorems

We present here two interposition theorems on multifunctions which are similar to the interposition theorem of Hahn on real-valued functions (see [4, p. 281]).

4.1. THEOREM. Suppose $\varphi, \psi \in M(X, Y)$, where Y is a separable metric space, φ is compact-valued and ψ is complete-valued, and let $\alpha > 0$. If $\varphi \in UB_\alpha$, $\psi \in LB_\alpha$ and $\varphi \in \psi$, then there exists a multifunction $\theta \in B_\alpha$ such that $\varphi < \theta < \psi$.

The above theorem does not hold in general for $\alpha = 0$. But we do have in that case the following theorem on convex-valued multifunctions. The weak continuity refers here to the continuity relative to the Vietoris topology generated by the weak topology of Y .

4.2. THEOREM. Suppose X is normal, Y is a separable reflexive normed vector space, $\varphi, \psi \in M_c(X, Y)$ and that one of

multifunctions $\{\varphi_n\}$ in B_ω .

2.5. THEOREM. Suppose X is normal, Y is a separable normed vector space, $\varphi \in M_{k,c}^*(X,Y)$ and that there exists a continuous multifunction ψ in $M_{k,c}(X,Y)$ such that $\varphi \subset \psi$. Then φ is USC iff it is the limit of a nonincreasing sequence of continuous multifunctions $\{\varphi_n\}$ in $M_{k,c}(X,Y)$.

3. Representation of multifunctions in B_α as limits of elementary multifunctions in lower Baire classes. Let us state first a general result on the Baire class of limits of sequences of multifunctions

3.1. THEOREM. Suppose Y is perfectly normal and let $\{\varphi_n\}$ be a sequence of multifunctions in $M(X,Y)$ which converges to φ .

(a) If $\varphi_n \in UB$ for each n , then $\varphi \in LB_{\alpha+1}$.

(b) If φ is compact-valued and $\varphi_n \in LB_\alpha$ for each n , then $\varphi \in UB_{\alpha+1}$.

(c) If φ is compact-valued, Y is second countable and $\varphi_n \in B_\alpha$ for each n , then $\varphi \in B_{\alpha+1}$.

The following theorem extends a well-known theorem on functions [5, p. 390] to multifunctions and it is easy to see that the result on functions is contained in this theorem.

3.2. THEOREM. Suppose Y is separable and metrizable, $\varphi \in M_k^*(X,Y)$ and that $\alpha > 1$. If $\varphi \in B_\alpha$, then φ is the limit of a sequence of elementary multifunctions $\{\varphi_n\}$ whose elements are in Baire classes lower than α .

Moreover, if $\alpha = \lambda + 1$ where λ is a limit ordinal, then the elements of each φ_n can be chosen to be in Baire classes lower than λ .

In the case when $\alpha = 1$ we have, on the other hand,

3.3. THEOREM. Suppose $\varphi \in M_k^*(X,Y)$, where X is normal and Y is a separable metrizable absolute retract (for metrizable spaces). Then $\varphi \in B_1$ iff it is the limit of a sequence of elementary multifunctions $\{\varphi_n\}$ whose elements are continuous.

The above theorem contains a similar result on functions

following conditions holds:

- (i) φ is compact-valued and Ψ is nonempty-valued,
- (ii) φ has nonempty weakly compact values and the dual space Y'' is separable.

If $\varphi \in \text{USC}$, Ψ is LSC and $\varphi \subset \Psi$, then there exists a weakly continuous multifunction $\theta \in M_c(X, Y)$ with weakly compact values such that $\varphi \subset \theta \subset \Psi$.

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