ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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WSP w Bydgoszczy

ON CLIQUISH FUNCTIONS

Let X,Y be topological spaces and let M be a metric space with metric d.

A function $f: X \to Y$ is said:
quasicontinous at a point $x_0 \in X$, if for every neighbourhood V of $f(x_0)$ and every neighbourhood U of x_0 there exists a nonempty open set $U_1 \subset U$ such that $f(U_1) \subset V$.;
cliquish at point $x_0 \in X$, if for every $\varepsilon > 0$ and every neighbourhood U of x_0 there exists a nonempty open set $U_1 \subset U$ such that $d(f(x'), f(x'')) < \varepsilon$ for $x', x'' \in U_1$.

Fudali proved:

Theorem 0. Let X be a Baire space, Y be a space that for each point $y \in Y$ there exists an open neighbourhood which satisfies the second countability axiom and let M be a matric space with a metric d. Further let $f: X \times Y \to M$ be a function such that for each $x \in X$ the section f_x is cliquish and for each $y \in Y$ the section f^y is quasicontinuous. Then f is cliquish.

I proved the following generalisation of Fudali's theorem:
Theorem 1. Let X,Y and M be this some as in theorem 0 and let $f: X \times Y \to M$ be a function such that for each $x \in X$ the section f_x is cliquish. Then f is cliquish if only if (a) for each $\varepsilon > 0$, the set $A_\varepsilon = \{(x,y) \in X \times Y;\ x \notin \mathrm{Cl}\,(\mathrm{Int}\ \{t \in X;\ d(f(t,y), f(x,y)) < \varepsilon\})\}$ is non dense (Int A and Cl A being interior and closure of the set A respectively).

Remark 1. Theorem 0 is contained in theorem 1. Then $A_\varepsilon = 0$ for each $\varepsilon > 0$.

Remark 2. There exists a real function $f: R^2 \to R$ such that for each $x \in X$ and $y \in Y$ the section f_x and f^y is cliquish

and f is not cliquish and the set $B(f)=\{(x,y)\in R^2;\ f^y$ is not quasicontinuous at $x\}$ is of first category.
For example, we give a characteristic function of a denumerable , dense set $A\subset R^2$ such that all sections A_x and A^y are empty or have one point.

I proved another theorems this some type:

Definition 1. Let S be a set of index and $f_s:X \longrightarrow M$ $(s\in S)$ be a family of functions. We said that the functions $f_s (s\in S)$ are equicliquish at a point $x\in X$ if for every $\varepsilon>0$ and for every neighbourhood $U\subset X$ of x there exists a non-empty open set $G\subset U$ such that for every $s\in S$ and every $x_1, x_2\in G$,

$$d(f_s(x_1),\ f_s(x_2))<\varepsilon .$$

Theorem 2. If all sections f_x of a function $f:X\times Y \longrightarrow M$ are equicliquish and all sections f^y are cliquish, then f is cliquish.

Theorem 3. Let X be such that for every $x\in X$ there exists an open neighbourhood which satisfies the second countability axiom. Let $f:X\times Y \longrightarrow R$ a function. If all sections f^y are cliquish and all sections f_x are increasing, then f is cliquish.

Remark 3. Let T_d be the density topology in R. There exists a function $f:R^2 \longrightarrow R$ such that all sections f^y are approximately continuous, all sections f_x are cliquish in T_d and f is not cliquish in $T_d \times T_d$.

Remark 4. If all sections f_x of a function $f:R^2 \longrightarrow R$ are upper semi equicontinuous (i.e. for every $\varepsilon>0$ and for every $y\in R$ there exists $\delta>0$ such that for every $x\in R$ and every $t\in(y-\delta,\ y+\delta)$, $f_x(t)-f_x(y)<\varepsilon$) and if all sections f^y are cliquish, then f is cliquish.
Let $X=Y=M=R$ and T be the topology of all sets of form $U-V$, where U is open and V is denumerable. There exists a function $f:R^2 \longrightarrow [0,1]$ such that all sections f_x are upper semi equicontinuous relative T and all sections f^y are cliquish relative T and f is not cliquish relative

T x T.

Remark 5. The family of all cliquish functions $f:R \to R$ relative the Euclidean topology is an algebra of functions. I proved that this is the smallest algebra of functions who include all quasicontinuous functions.

Theorem 4. Every cliquish function $f:R \to R$ is the sum of four quasicontinuous functions.

Remarque 6. There exists a function cliquish $f:R \to R$ (even of Baire 1 class) who is not a finite product of quasicontinuous functions. For example,

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ and } (p,q) = 1 \\ 0 & \text{if } x \text{ is not rationnel .} \end{cases}$$

Remark 7. Every derivative $f:R \to R$ is cliquish function. Analogy every partial derivative f'_x or f'_y of continuous function $f:R^2 \to R$ cliquish function. A partial derivative f'_x of discontinuous function perhap not be cliquish function. For example, if a function $g:R \to R$ is not cliquish, the partial derivative f'_x of the function $f(x,y) = x \cdot g(y)$ is not cliquish. Davies proved that the partial derivative f''_{xy} of function $f:R^2 \to R$ is Baire's class 2 and he proved that there exists a partial derivative $f''_{xy} = f''_{xy}$ of function $f:R^2 \to R$ such that f''_{xy} is not Baire's 1 class. There exists a partial derivative $g = f''_{xy} = f''_{xy}$ of function $f:R^2 \to R$ who is not cliquish at any point such that all sections g_x and g^y are approximately continuous Simultaneous every function $f:R^2 \to R$ such that all sections f_x and f^y are approximately continuous and almost everywhere continuous is cliquish.

Finished we give an partially answer to the problem of Petruska.

Problem. Is there a function f such that f'_y and f''_{xy} exists everywhere while f''_{yx} does not exist at any point.

Theorem 5. If a function $f:R^2 \to R$ is such that the partial derivatives f'_y and f''_{xy} exists everywhere and the partial derivative f''_{xy} is bounded in an closed interval $[a,b] \times [c,d]$, then the partial derivative f''_{yx} exist and is equal f''_{xy} almost everywhere in $[a,b] \times [c,d]$.