ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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ON POINTS OF THE APPROXIMATE SEMICONTINUITY

We use the following notation. The sign $\mathcal{J}$ denotes the σ-ideal of the measure zero subsets of R. If $X \subseteq R$ is measurable then $\varphi(X)$ denotes the set of all density points of X.

For the function $f:R \to R$ the signs $\text{ap-}\lim\inf_{t \to x} f(t)$ and $\text{ap-}\lim\sup_{t \to x} f(t)$ denote the approximately lower and upper limit of f at x, respectively. Notice that

$$\text{ap-}\liminf_{t \to x} f(t) = \sup\{y : D_s^*(x,\{t: f(t) < y\}) = 0\} \text{ and}$$

$$\text{ap-}\limsup_{t \to x} f(t) = \inf\{y : D_s^*(x,\{t: f(t) > y\}) = 0\}, \text{ where}$$

$$D_s^*(x,A) = \lim_{n \to \infty} \sup\left\{\frac{m^*(A \cap I)}{m(I)} : m(I) < \frac{1}{n}\right\} .$$

The signs $A(f)$, $S_a(f)$, $S_a^1(f)$ denote the sets of all points at which f is approximate continuous, upper and lower semicontinuous, respectively:

$$A(f) = \{x \in R: f(x) = \text{ap-}\liminf_{t \to x} f(t) = \text{ap-}\limsup_{t \to x} f(t)\},$$

$$S_a(f) = \{x \in R: f(x) \geq \text{ap-}\limsup_{t \to x} f(t)\},$$

$$T_a(f) = \{x \in R: f(x) > \text{ap-}\limsup_{t \to x} f(t)\},$$

$$S_a^1(f) = \{x \in R: f(x) \leq \text{ap-}\liminf_{t \to x} f(t)\},$$

$$T_a^1(f) = \{x \in R: f(x) < \text{ap-}\liminf_{t \to x} f(t)\}.$$

Z. Grande showed in [1] the following facts.

FACT 0. For every function $f:R \to R$ the set $A(f)$ is measurable.

FACT I. For every $f:R \to R$ we have $T_a(f) \cup T_a^1(f) \in \mathcal{J}$.

FACT 2. The sets $S_a(f) - A(f)$ and $S_a^1(f) - A(f)$ do not contain measurable sets of the positive measure.

FACT 3. Let A,B,C are subsets of R such that

- $C \in \mathcal{J}$,
- $B \subseteq A$ and $C \subseteq A-B$,
- there exists a G_δ set D such that $B = D-C$,
- the set $A-B$ do not contain a measurable sets of positive measure,
- $R-D$ is the sum $M \cup N$, M and N are a F_σ sets, $N \in \mathcal{J}$ and $M \subseteq \varphi(M)$.

Then there exists a function $f:R \to R$ such that $A(f) = B$, $S_a(f) = A$, and $T_a(f) = C$.

My results are following (see [2]).

FACT 4. (MA) Let A, C, A' and C' are subsets of R such that

(i) $C \cup C' \in \mathcal{J}$,

(ii) $B = A \cap A'$,

(iii) $C \subseteq A-B$ and $C' \subseteq A'-B$,

(iv) there exists a G_δ set D such that $B = D-(C \cup C')$,

(v) the sets $A-B$ and $A'-B$ do not contain a measurable sets of the positive measure,

(vi) there exists a sequence $(G_n)_{n \in N}$ of open sets such that $G_{n+1} \subseteq G_n$, $D = \bigcap_{n \in N} G_n$ and $\bigcap_{n \in N} \varphi(G_n) - B$ is a F_σ set .

Then

(x) there exists a function $f:R \to R$ such that $A(f) = B$, $S_a(f) = A$, $T_a(f) = C$, $S_a^1(f) = A'$ and $T_a^1(f) = C'$.

FACT 5.(MA) Let A, B, C, A' and C' are subsets of R and the conditions (i)-(v) and

(vii) $R-D$ is the sum $M \cup N$, M and N are a F_σ sets, $N \in \mathcal{J}$ and $M \subseteq \varphi(M)$

hold. Then the statement (x) holds too .

REMARK 0. None of the implications (vi) $\Rightarrow$ (vii) and (vii) $\Rightarrow$ (vi) holds.

We consider the following examples.

EXAMPLE 0. Let $C \subseteq \langle 0,1 \rangle$ be the Cantor's set such that $C \in \mathcal{J}$ and P be the set of all birateral limit points of

C. Since the set C-P is dense in C and P is of the second category in C, P is not F_σ set.
Let us assume that $B = R-P$. Then the condition (vi) does not hold. If D is a G_δ set and $B \subseteq D$ then $R-D \subseteq P$ and $R-D \in \mathcal{J}$. Thus for $M = \emptyset$ and $N = R-D$ we have $R-D = M \cup N$.
So the condition (vii) holds.

EXAMPLE 1. Let C be the Cantor's set such that $C \notin \mathcal{J}$ and $C \subseteq \langle 2,3 \rangle$, C' be the set of all birateral limit points of C and $C'' = C-C'$. It is clear that $\varphi(C) \subseteq C'$.
If $M \subseteq C'$ is a F_σ set then $N = C-M$ is a G_δ set and N is dense in C. Hence N is residual in C.

Suppose that N is a F_σ set. Then $N = \bigcup_{n \in N} F_n$ and are closed and pairwise disjoint sets [3]. Since N is of the second category in C, there exists F_n which is of the second category in C. Since F_n is closed, there exists an open interval I such that $\emptyset \neq C \cap I \subseteq F_n$. Hence $F_n \notin \mathcal{J}$.
Assume that $B = D = R-C$. Then the condition (vii) does not hold.
Let $G_n = D$ for $n = 1,2,\ldots$. Notice that $\bigcap_{n \in N} \varphi(G_n) - B = \emptyset$.
Hence the condition (vi) holds.

REMARK 1. There exists a set B and there exists a function $f:R \to R$ such that $A(f) = B$ and the conditions (vi) and (vii) do not hold.
We consider the following example.

EXAMPLE 2. Let P be the set defined in Example 0, C be the Cantor's set from Example 1 and $B = R-(C \cup P)$.
Let $g:R \to R$ be a function such that $A(g) = R-P$ and $h:R \to R$ be a function such that $A(h) = R-C$.
Let us define a function $f:R \to R$ as follows $f(x) = g(x) + h(x)$.
It is easy to show that $A(f) = B$ and the conditions (vi) and (vii) do not hold.

PROBLEMS

(1) Let us assume that for $A, A', B, C, C' \subseteq R$ the conditions (i)-(v) hold. Does then the statement (x) hold ?

(2) Is there for every function $f:R \to R$ a G_δ set D such that $A(f) = D - (T_a(f) \cup T_a^{1}(f))$?

REFERENCES

[1] Grande Z., Quelques remarques sur la semicontinuité superieure, Fund. Math. CXXV/1/1985/

[2] Natkaniec T., Zbiory punktów ciągłości i półciągłości funkcji rzeczywistych, doctor's thesis

[3] Sierpiński W., Sur une propriété des ensembles F_σ linéaires, Fund. Math. 14 (1929)