ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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ON CONTINUOUS AND LOCALLY NON-CONSTANT FUNCTIONS

1. Introduction

Here and in the sequel, (X,d) is a metric space, $(Y,\|\cdot\|)$ is a normed space, $C(X,Y)$ is the space of all continuous functions from X into Y endowed with the metric $\varrho(f,g) = \min\{\sup \|f(x) - g(x)\|, 1\}$, $\mathcal{R}(X,Y)$ is the set of all $f \in C(X,Y)$ such that $(\operatorname{int} f^{-1}(y)) = \emptyset$ for every $y \in Y$. Any function $f \in \mathcal{R}(X,Y)$ is said to be locally non-constant. If $Y = \mathbb{R}$, we put $C(X) = C(X,Y)$ and $\mathcal{R}(X) = \mathcal{R}(X,Y)$.

The aim of this paper is to give a contribution to the study of the class $\mathcal{R}(X,Y)$. In particular, we prove that $\mathcal{R}(X,Y)$ is dense in $C(X,Y)$ provided that either X is locally connected and $\mathcal{R}(X) \neq \emptyset$ or X is separable and perfect and Y is complete. Also, we obtain some characterizations of the connectedness and of the sepaparability of X in terms of the class $\mathcal{R}(X)$. Several further properties of the functinns belonging to $\mathcal{R}(X)$ can be found in [1] and [2].

2. Two density theorems

We begin this section by proving the following result:

THEOREM 2.1. Suppose that X is locally connected and that $\mathcal{R}(X) \neq \emptyset$. Then, for every $f \in C(X,Y)$ and every $\varepsilon > 0$, there exists $f_\varepsilon \in \mathcal{R}(X,Y)$ such that $\varrho(f_\varepsilon, f) \leq \varepsilon$. Moreover, $f_\varepsilon(X) \subseteq \operatorname{conv}(f(X))$ provided that f is not constant.

PROOF. As it is not restrictive, we assume $\varepsilon < 1$.

Choose $\lambda \in \mathcal{R}(X)$ and put $\psi(x) = \dfrac{\pi + 2\operatorname{arctg}(\lambda(x))}{2\pi}$ for every $x \in X$. Obviously, $\psi \in \mathcal{R}(X)$ and $\psi(X) \subseteq]0,1[$.

At first, we prove our theorem in the case where X is also connected and f is not constant. Without loss of generality, we can suppose $\operatorname{diam}(X) \leq 1$. Let $\tilde{Y} = \{y \in f(X) : \operatorname{int}(f^{-1}(y)) \neq \emptyset\}$. If $\tilde{Y} = \emptyset$, the thesis is obvious. Therefore, suppose that $\tilde{Y} \neq \emptyset$ and, for every $y \in \tilde{Y}$, put $A_y = \operatorname{int}(f^{-1}(y))$. Consider now the function $g : X \to \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases} d(x, \partial A_y) & \text{if } x \in A_y,\ y \in \tilde{Y} \\ 0 & \text{if } x \in X \setminus \bigcup_{y \in \tilde{Y}} A_y, \end{cases}$$

where, as usual, $d(x, A_y) = \inf\{d(x,\nu) : \nu \in \partial A_y\}$ (observe that, by the actual assumptions, ∂A_y is non-empty for every $y \in \tilde{Y}$). Let us show that g is continuous. Tho this end, let $x^* \in X$ and $\sigma > 0$. If $x^* \in X \setminus \partial \bigcup_{y \in \tilde{Y}} A_y$, then the continuity of g at x^* is obvious. Therefore, suppose that $x^* \in \partial \bigcup_{y \in \tilde{Y}} A_y$. Then $g(x^*) = 0$. Let Ω be a connected, open neighbourhood of x^* such that $\operatorname{diam}(\Omega) < \sigma$. Let x be any point of Ω. If $x \notin \bigcup_{y \in \tilde{Y}} A_y$, we have $g(x) = 0$. On the other hand, if $x \in A_y$, $y \in \tilde{Y}$, by the connectedness of Ω, we have $\partial A_y \cap \Omega \neq \emptyset$. Hence, if we choose $\bar{x} \in \partial A_y \cap \Omega$, we have $g(x) = d(x, \partial A_y) \leq d(x, \bar{x}) \leq \operatorname{diam}(\Omega) < \sigma$. Therefore, g is continuous at x^*.

Now, denote by φ the Lebesgue singular function on $[0,1]$. Put $h(x) = \varphi(g(x))$ for every $x \in X$. Let us show that for every non-empty, open set $\Omega \subseteq \bigcup_{y \in \tilde{Y}} A_y$, there exists another non-empty, open set $\Omega^* \subseteq \Omega$ such that $h_{|\Omega^*}$ is constant and positive. Indeed, let $\bar{y} \in \tilde{Y}$ be such that $\Omega \cap A_{\bar{y}} \neq \emptyset$ and let Ω' be a non-empty, open, connected subset of $\Omega \cap A_{\bar{y}}$. Obviously, we have $g(x) > 0$ for every $x \in \Omega'$. If $g_{|\Omega'}$, is constant, then we can take $\Omega^* = \Omega'$. In the opposite case, $g(\Omega')$ is a (non-degenerate) interval contained in $]0,1]$. Taking into account the definition of φ, we get the existence of an open interval $J \subseteq g(\Omega')$ such that $\varphi_{|J}$ is constant. At this point, it suffices to take $\Omega^* = g^{-1}(J) \cap \Omega$.

For every fixed $\bar{y} \in f(X)$, consider now the function

$f_{\varepsilon,\bar{y}} : X \to Y$ defined as follows:

$$f_{\varepsilon,y}(x) = f(x) + \varepsilon \frac{\psi(x)\,h(x)}{1 + \|\bar{y} - f(x)\|}(\bar{y} - f(x)) \quad \text{for every } x \in X.$$

Obviously $f_{\varepsilon,\bar{y}}$ is continuous, $\rho(f_{\varepsilon,\bar{y}}, f) \le \varepsilon$ and $f_{\varepsilon,y}(X) \subseteq \subseteq \mathrm{conv}(f(X))$. Observe also that $f_{\varepsilon,y}(x) = f(x)$ for every $x \in X \setminus \bigcup_{y \in \tilde{Y}} A_y$.

Let us show that if Ω is an open subset of X such that $\Omega_1 = \Omega \setminus \overline{A_{\bar{y}}} \neq \emptyset$, then $f_{\varepsilon,y|\Omega_1}$ is not constant. Indeed, assume first $\Omega_1 \setminus \overline{\bigcup_{y \in \tilde{Y}} A_y} \neq \emptyset$. Then we have $f_{\varepsilon,y}(x) = f(x)$ for every $x \in \Omega_2 = \Omega_1 \setminus \overline{\bigcup_{y \in Y} A_y}$ and so $f_{|\Omega_2}$ is not constant, since, otherwise, we would have $\Omega_2 \subseteq A_y$ for some $y \in \tilde{Y}$. Assume now $\Omega_1 \subseteq \bigcup_{y \in Y} A_y$. Then $\Omega_1 \cap A_{y^*} \neq \emptyset$ for some $y^* \in \tilde{Y}$, $y^* \neq \bar{y}$. Let $\Omega_3 = \Omega_1 \cap A_{y^*}$. Then

$$f_{\varepsilon,\bar{y}}(x) = y^* + \varepsilon \frac{\psi(x)\,h(x)}{1 + \|\bar{y} - y^*\|}(\bar{y} - y^*) \quad \text{for every } x \in \Omega_3.$$

As we have seen above, there exist a non-empty open set $\Omega_3^* \subseteq \Omega_3$ and a positive number c such that $h(x) = c$ for every $x \in \Omega_3^*$. On the other hand, since $\psi \in \mathcal{R}(X)$, there exist two points $x', x'' \in \Omega_3^*$ such that $\psi(x') \neq \psi(x'')$. We have

$$f_{\varepsilon,y}(x') - f_{\varepsilon,y}(x'') = \frac{\varepsilon c(\psi(x') - \psi(x''))}{1 + \|\bar{y} - y^*\|}(\bar{y} - y^*) \neq 0$$

and so $f_{\varepsilon,\bar{y}|\Omega_1}$ is not constant.

Now observe that if $f(X) \setminus \tilde{Y} \neq \emptyset$, then, if we choose $\bar{y} \in f(X) \setminus \tilde{Y}$ and put $f_\varepsilon = f_{\varepsilon,y}$, the function f_ε satisfies the required properties.

Nevertheless, it may happen that $\tilde{Y} = f(X)$ see Example 3.1. In this case, choose $y', y'' \in f(X)$, $y' \neq y''$, and put:

$$f_\varepsilon(x) = \begin{cases} f_{\varepsilon,y'}(x) & \text{if } x \in X \setminus A_{y'}, \\ f_{\varepsilon,y''}(x) & \text{if } x \in A_{y'}. \end{cases}$$

It is obvious that $\rho(f_\varepsilon, f) \le \varepsilon$ and that $f_\varepsilon(X) \subseteq \mathrm{conv}(f(X))$.

Moreover, the continuity of f_ε follows immediately from the fact that $f_{\varepsilon,y'}(x)= f(x)= f_{\varepsilon,y''}(x)$ for every $x \in \partial A_{y'}$.
Finally, let us show that $\mathrm{int}\,(f^{-1}(y))\neq \emptyset$ for every $y \in f(X)$. To this aim, let Ω be any non-empty, open subset of X. If $\Omega'' = \Omega \cap \overline{A_{y'}} \neq \emptyset$, then $f_{\varepsilon|\Omega''} = f_{\varepsilon,y'|\Omega''}$ and we have proved that $f_{\varepsilon,y'|\Omega''}$ is not constant. On the contrary, if $\Omega\subseteq \overline{A_{y'}}$, then, by continuity of f, $f(\Omega)=\{y'\}$ and so $\Omega \subseteq A_{y'}$. Hence $f_{\varepsilon|\Omega} = f_{\varepsilon,y''|\Omega}$ and $f_{\varepsilon,y''|\Omega}$ is not constant since $\Omega \cap A_{y''}= \Omega \neq \emptyset$. Therefore, under the additional assumption that X is connected and that f is not constant, our theorem is proved.

Finally, let us consider the general case.

If f is constant, then we take f_ε defined as follows: $f_\varepsilon(x)= f(x) + \varepsilon \psi(x)\bar{y}$ for every $x \in X$, being $\bar{y} \in Y$ such that $\|\bar{y}\| = 1$.

So, let f be non-constant. Denote by $\mathcal{F}$ the family of all connected components of X. Since X is locally connected, every member of $\mathcal{F}$ is open and hence locally connected. Let $\mathcal{F}^* = \{K \in \mathcal{F} : f_{|K}$ is constant$\}$ and for every $K\in \mathcal{F}^*$, choose $y_K \in f(X)$ such that $f(K)\neq \{y_K\}$. Moreover, for every $K\in\mathcal{F}\setminus\mathcal{F}^*$, let $f_{\varepsilon,K}$ be a continuous and locally non-constant function from K into Y such that $\sup_{x\in K} \|f_{\varepsilon,K}(x) - f(x)\| \leq \varepsilon$ and $f_{\varepsilon,K}(K)\subseteq \mathrm{conv}\,(f(K))$. Such a function there exists by the first part of the proof. Now let $\bar{y}\in Y$, $\|\bar{y}\| = 1$, and consider the function $f_\varepsilon : X \to Y$ defined as follows:

$$f_\varepsilon(x)= \begin{cases} f(x)+ \dfrac{\varepsilon\psi(x)}{1 + \|y_K-f(x)\|}(y_K - f(x)) & \text{if } x\in K,\ K\in\mathcal{F}^* \\ f_{\varepsilon,K}(x) & \text{if } x\in K,\ K\in \mathcal{F}\setminus\mathcal{F}^* \end{cases}$$

The function f_ε has the desired properties and so our theorem is completely proved.

The other density result we want to present here is the following:

THEOREM 2.2. Suppose that X is separable and perfect and that Y is complete. Then $\mathcal{R}(X,Y)$ is a dense $\mathcal{G}_\delta$-set in $C(X,Y)$.

PROOF. Let $\{\Omega_n\}$ be a sequence of non-empty open subsets of X, which is a base for X. For every $n \in \mathbb{N}$, put $D_n = \{f \in C(X,Y) : f_{|\Omega_n}$ is not constant$\}$. Obviously the set D_n is open in $C(X,Y)$. We claim that it is dense here. Indeed, let $f \in C(X,Y) \setminus D_n$ and $\varepsilon > 0$. Choose $x^* \in \Omega_n$, $y \in Y$, $\|y\| = 1$, and put $f_\varepsilon(x) = f(x) + \frac{\varepsilon d(x,x^*)}{1 + d(x,x^*)}\, y$ for every $x \in X$. As $f_{|\Omega_n}$ is constant and Ω_n is infinite owing to the perfectness of X, we have $f_\varepsilon \in D_n$. Moreover $\varrho(f_\varepsilon, f) \leq \varepsilon$ and so the claim is proved.

Since $C(X,Y)$ is a complete metric space, the set $\bigcap_{n=1}^{\infty} D_n$ is dense in $C(X,Y)$. On the other hand, we have $\mathcal{R}(X,Y) = \bigcap_{n=1}^{\infty} D_n$. This completes the proof.

3. Further results and applications

In this section we present some consequences of Theorems 2.1 and 2.2 and establish some characterizations of the connectedness and of the separability of the space X in terms of continuous and locally non-constant real functions on X. But before, we want to display Example 3.1 below which has been quoted in the proof of Theorem 2.1.

EXAMPLE 3.1. Suppose that X is a (connected, locally connected, with $\mathcal{R}(X) \neq \emptyset$) metric space which contains a set X^*, having the continuum power, such that $\inf\{d(x,\nu) : x, \nu \in X^*, x \neq \nu\} > 0$ for instance: $X = L^\infty([0,1])$, $X^* = \{1_{[0,t]} : t \in]0,1]\}$). Then, there exists $f \in C(X)$ such that $\operatorname{int}(f^{-1}(y)) \neq \emptyset$ for every $y \in \mathbb{R}$.

Indeed, let $\varphi : X^* \to \mathbb{R}$ be onto. Put $\alpha = \inf\{d(x,\nu) : x, \nu \in X^*, x \neq \nu\}$ and $A = \bigcup_{x \in X^*} \bar{B}(x, \frac{\alpha}{3})$, where $\bar{B}(x, \frac{\alpha}{3}) = \{\nu \in X : d(x,\nu) \leq \frac{\alpha}{3}\}$. Plainly, the set A is closed.

Consider now the function $g : A \to \mathbb{R}$ defined as follows:

$$g(x)=\varphi(\nu) \quad \text{if} \quad x\in \bar{B}(\nu,\tfrac{\alpha}{-}) \ , \quad \nu\in X^* \ .$$

Obviously, g is continuous and $g(A)=\mathbb{R}$.

By the Tietze extension theorem, there exists $f\in C(X)$ such that $f_{|A}=g$. Clearly, $\mathrm{int}\,(f^{-1}(y))\neq\emptyset$ for every $y\in\mathbb{R}$. This proves the above claim.

Now, we recall that a $f\in C(X)$ is said to be inductively--open (in the Arhangel'skii sense) if there exists $X'\subseteq X$ such that $f(X')=f(X)$ and that $f : X'\to f(X)$ is an open function.

By applying jointly Theorem 2.1 and Theorem 2.4 of [1], we obtain the following result:

THEOREM 3.1. Suppose that X is connected and locally connected and that $\mathcal{R}(X)\neq\emptyset$. Then, every continuous real function on X is the uniform limit of a sequence of continuous, inductively open, real functions on X .

It would be interesting to know whether, under the hypotheses of Theorem 3.1, the following more general thesis holds: for every non-constant $f\in C(X)$ there exist a sequence $\{f_n\}$ in $\mathcal{R}(X)$, which converges uniformly to f , and a set $X'\subseteq X$, such that $f_n(X')=f(X)$ and that $f_n : X'\to f(X)$ is an open function for every $n\in\mathbb{N}$.

As an application of Theorem 2.2 we present the following genericity result for implicit differential equations.

THEOREM 3.1. Let $a,b>0$ and X be a closed real interval. Let $\mathcal{M}$ be the set of all $g\in C(X)$ such that for ever $f\in C([0,a]\times[-b,b])$ satisfying $f([0,a]\times[-b,b])\subseteq g(X)$ the implicit Cauchy problem

$$\begin{cases} g(x)=f(t,x) \\ x(0)=0 \end{cases}$$

has at least a Lipschitzian local solution. Then, $\mathcal{M}$ is a residual set in $C(X)$.

PROOF. By Theorem 1.2 of [3] , we have $\mathcal{R}(X)\subseteq\mathcal{M}$. Then our thesis follows from Theorem 2.2.

Observe that Theorem 3.2 provides a partial answer to Problem 2.4 of [3] .

To state the next result we introduce some notations. For every $f \in C(X)$, we put $E_f = \{x \in X : x$ is a relative, non-absolute, extremum point for $f\}$. Also, we put $(X) = \{f \in C(X): f(X) = f(X \setminus E_f)\}$.

THEOREM 3.3. Let $\mathcal{R}(X) \neq \emptyset$. Then, the following are equivalent:

1) X is connected.

2) $\mathcal{R}(X) \subseteq (X)$.

Proof. The implication 1) $\Longrightarrow$ 2) follows directly from Theorem 5.1 of [4] .

Let us show that 2) $\Longrightarrow$ 1) . Assume that X is not connected. Let X_1, X_2 be two non-empty, open subsets of X such that $X_1 \cup X_2 = X$, $X_1 \cap X_2 \neq 0$. Let $f \in \mathcal{R}(X)$ and $x_1 \in X_1$, $x_2 \in X_2$. Consider the function $g : X \longrightarrow \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases} |f(x) - f(x_1)| & \text{if} \quad x \in X_1 \\ -|f(x) - f(x_2)| & \text{if} \quad x \in X_2 \end{cases}$$

It is easy to check that $g \in \mathcal{R}(X)$. On the other hand, $0 \in g(X)$ but, if $x^* \in g^{-1}(0)$, then, necessarily, $x^* \in E_g$. Hence $g \notin \mathcal{E}(X)$. So we get a contradiction.

Observe that in the proof of Theorem 3.3 the metrizability of X has no role.

The last result of this paper is the following:

THEOREM 3.4. Suppose that X is locally connected. Then, the following are equivalent:

1) X is separable and perfect.

2) There exist $f \in \mathcal{R}(X)$ and $D \subseteq f(X)$, D dense in $f(X)$, such that $f^{-1}(y)$ is separable for every $y \in D$.

PROOF. The implication 1 $\Longrightarrow$ 2 follows directly from Theorem 2.2.

Let us show that 2) $\Longrightarrow$ 1 . The perfectness of X follows directly from the faot that $X \neq \emptyset$. To prove that X is separable, we can suppose, without loss of generality, that D is countable. For every y D , choose a countable, dense subset X_y of $f^{-1}(y)$. Consider the countable set $X^* = \bigcup_{y \in D} X_y$. We claim that X is dense in X . To show this, let $\bar{x}$ be any point of X and Ω be any open neighbourhood of x . Since X is locally connected and $f \in \mathcal{R}(X)$ the set $f(\Omega)$ contains a (non-degenerate) interval. Let $y^* \in f(\Omega) \cap D$. By the density of X_{y^*} in $f^{-1}(y^*)$, it follows that $\Omega \cap X_{y^*} \neq \emptyset$ and so, a fortiori, $\Omega \cap X^* \neq \emptyset$.

Observe that in Theorem 3.4 to prove that 2) $\Longrightarrow$ 1), the metrizability of X is superfluous, as well as, to prove that 1) $\Longrightarrow$ 2), the local connectedness of X.

To conclude we mention two questions related to the matter treated in this paper. They seem to be open.

PROBLEM 3.1. Does Theorem 2.1 hold without the assumption that X is locally connected ?

PROBLEM 3.2. Does a connected, locally connected, perfect metric space X exist such that $\mathcal{R}(X) \neq \emptyset$?

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Sunto

Si studiano alcune proprietà delle funzioni continue e localmente non constanti. In particolare, si stabiliscono due teoremi sulla densità dell' insieme di tali funzioni nello

spazio delle funzioni continue, munito della topologia della convergenza uniforme. Si danno inoltre, mediante tali funzioni, alcune caratterizzazioni della connessione e della separabilità di uno spazio.

Note.

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