ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY

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# MINIMAL PERIODIC ORBITS OF MAPPINGS OF AN INTERVAL

## 1. Introduction.

Let I denote a compact interval on the real line and let $C^o(I,I)$ denote the space of continuous maps from I into itself. Let N denote the set of positive integers. For any $n \in N$ we define $f^n$ inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. Let $f^o$ denote the identity map of I. A point $x \in I$ is said to be a periodic point of $f$ if $f^n(x) = x$ for some $n \in N$. In this case the smallest element of $\{n \in N : f^n(x) = x\}$ is called the period of x. We define the orbit of x to be $\{f^n(x)\, n=0,1,2\}$. If x is a periodic point we say the orbit of x is a periodic orbit, and we define the period of the orbit to be the period of x. Clearly, if x is a periodic point of f of period n, then the orbit of x contains n points and each of these points is a periodic point of f of period n.

Let a,b be real numbers and let A,B be subsets of the real line. We denote $f(a)= b$ and $f(A)= B$ by $a \xrightarrow{f} b$ and $A \xrightarrow{f} B$, respectively. Similarly, $A \xleftarrow{f} B$ means $f(B)= A$ and $A \xleftrightarrow{f} B$ means $f(A)= B$ and $f(B)= A$. Finally, $f|M$ denotes the restriction of f to the set M.

THEOREM (A.N. Sarkovskii, see [2] or [3]). Let $f \in C^o(I,I)$. Let us consider the following ordering of the positive integers $3,5,7,\ldots,2\cdot3,2\cdot5,2\cdot7,\ldots,4\cdot3,4\cdot5,\ldots,8\cdot3,8\cdot5,\ldots,\ldots,8,4,2,1$ Let f have a periodic orbit of period n. If m is to the right of n (in the above ordering), then f has a periodic orbit of period m.

It is known that for every n there exists a function f such that f has a periodic orbit of per[illegible] m if and only if $m$ is not to the left of $n$. Similarly, there exists a function $f$ such that $f$ has a periodic orbit of period $m$ if and only if $m$ is a power of 2.

DEFINITION. A periodic orbit $P$ of $f$ of period $n$ is a minimal periodic orbit of $f$, if $f$ has no periodic orbits of periods less (in Šarkovskiǐ sense) than $n$.

DEFINITION. We say that a periodic orbit $P$ of $f$ is potentially minimal if there exist a compact interval $I \supset P$ and a continuous function $g$ from $I$ into itself with the following two properties:

(i) $f|P = g|P$

(ii) $P$ is a minimal periodic orbit of $g$.

It is possible that for some $P$, $f_1$, $f_2$ the set $P$ is a periodic orbit both of $f_1$ and $f_2$ and the periodic orbit $P$ of $f_1$ is potentially minimal but the periodic orbit $P$ of $f_2$ is not potentially minimal.

Similarly, it is possible that $P$ is a periodic orbit both of $g_1$ and $g_2$, $g_1 | P = g_2|P$, $P$ is a minimal periodic orbit of $g_1$ and $P$ is not a minimal periodic orbit of $g_2$.

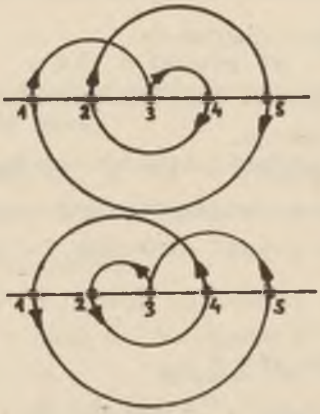
The main problem connected with minimal periodic orbits is the following.

PROBLEM. Characterize potentially minimal orbits. (Clearly, without loss of generality we may solve this problem only for periodic orbits of the form $\{1,2,\dots,n\}$. Hence, let us assume that $f$ has a periodic orbit $\{1,2,\dots,n\}$ and investigate under which assumptions this orbit is potentially minimal.)

RESULTS

1) case $n = 2p + 1$, $p \in N$

P. Štefan [3] has proven that there are exactly two types of potentially minimal orbits of period $2p+1$. They have "spiral" structure (see Fig. 1).




$n=5$
(similarly for $n = 3,7,9\ldots$)

Fig. 1

2) case $n = 2^m$ , $m \in N$

From Theorem A in [1] it follows a necessary condition for a periodic orbit $P$ of $f$ of period $2^m$ to be potentially minimal. This necessary condition is the following:

For any subset $\{q_1,\ldots,q_k\}$ of $P$ where $k$ divides $2^m$ and $k \geq 2$ , and any positive integer $r$ which divides $2^m$ , such that $\{q_1,\ldots,q_k\}$ is periodic orbit of $f^r$ with $q_1 < q_2 < \ldots < q_k$ , we have

$$f^r(\{q_1,\ldots,q_{k/2}\}) = \{q_{k/2+1},\ldots,q_k\} .$$

3) case $n = 2.(2p + 1)$ , $p \in N$ (Ľ.Snoha 1983)

Let $L = \{1,\ldots,2p+1\}$, $R = \{2p+2,\ldots,2.(2p+1)\}$. Let $P = L \cup R$ be a periodic orbit of $f$ of period $2.(2p+1)$. The following conditions play an important role in the characterization.

NC Periodic orbit $P$ of $f$ of period $2.(2p+1)$ is such that

(a) $L \overset{f}{\longleftrightarrow} R$

(b) L and R are minimal (in Štefan sense) periodic orbits of the function $f^2$ .

(1) f is monotonic on L or on R.

(2) Four numbers 1,2p+1, 2p+2, 2.(2p+1) are "neighbours" in the periodic orbit P. This means, that there exists such a permutation $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of the set $\{1, 2p+1, 2p+2, 2.(2p+1)\}$, that

$$\alpha_1 \xrightarrow{f} \alpha_2 \xrightarrow{f} \alpha_3 \xrightarrow{f} \alpha_4$$

(3) $\{1, 2p+1\} \xrightarrow{f} \{2p+2, 2(2p+1)\}$ or $\{1, 2p+1\} \xleftarrow{f} \{2p+2, 2(2p+1)\}$

THEOREM.

(i) Periodic orbit $\{1,2,3,4,5,6\}$ of f period 2.3 is potentially minimal if and only if (NC)(a) is true.

(ii) Periodic orbit $\{1,\ldots,2(2p+1)\}$, $p>1$ of f of period $2(2p+1)$ is potentially minimal if and only if at least one of the following conditions is satisfied:

(NC) and (1) ,(NC) and (2),(NC) and (3) .

Consequently, there exist 12 types of potentially minimal orbits of period 2.3 and 8 types of potentially minimal orbits of period $2(2p+1)$, $p>1$ .

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[3] Stefan P., A theorem of Sarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line, Comm. Math. Phys. 54 (1977), 237-248