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CONCERNING A.E. - CONTINUOUS EXTENSIONS OF BAIRE 1 FUNCTIONS

This paper presents new results concerning extensions of Baire 1 vector-valued functions defined on a subset of finite-dimensional euclidean space to finely continuous [15] (e.g. a.e. - continuous, cf. [22]) functions defined on the whole space. In particular a problem 13 posed by prof. Z. Grande in [9] is solved here. In order to prove our extension theorem the notion of z- lower semicontinuous multifunction is introduced and the theory of continuous selectors for such multifunctions is developed. Let R denotes the real line and $C^*(X)$ the lattice of continuous, bounded and real-valued functions defined on the topological space X.
The following general insertion theorem is stated in [13] (see also [3], [12], [14], [24], [26] for related results):

THEOREM 0 ([13]). Let X be an arbitrary topological space and let L(X) and U(X) be classes of bounded functions defined on X such that any constant function is in the intersection $L(X) \cap U(X)$ and such that if $g \in U(X)$, $f \in L(X)$ and $r \in R$ then $g \wedge r \in U(X)$ and $f \vee r \in L(X)$. The following statements are equivalent:

(i) If $f \in L(X)$, $g \in U(X)$ and $g \leq f$, then there exists a function h belonging to the lattice $C^*(X)$ such that $g \leq h \leq f$ and such that $g(x) < h(x) < f(x)$ for each x for which $g(x) < f(x)$.

(ii) If $f \in L(X)$, $g \in U(X)$ and $r \in R$, the Lebesgue sets

(1) $L_r(f) := \{x \in X : f(x) \leq r\}$ and $L^r(g) := \{x \in X : g(x) \geq r\}$ are zero sets in X.

(iii) If $f \in L(X)$ and $g \in U(X)$, then f (respectively g) is the pointwise limit of an increasing (resp. decreasing) sequence of continuous functions.

Recently similar result has been independently reproved in [27],[2] . It is also easily observed that the lattice $C^*(X)$ in theorem 0 may be replaced by others linear lattices of functions (see [19],[20],[21] in that direction). In the situation where $U(X)$ and $L(X)$ are the classes of upper and lower semicontinuous functions resp., the equivalence of (i) and (ii) is due to Michael [18], the equivalence of (ii) and (iii) is due to Tong [26] , and each of the conditions being equivalent to X is perfectly normal.

It is noted ([13] , [14]), that the boundedness condition placed on the functions in Theorem 0 causes no loss in generality if the properties that define the classes $L(X)$ and $U(X)$ are preserved under an order preserving homeomorphism from R onto a bounded interval.

Let us recall that a function $f : X \to R$ is z- lower semicontinuous (resp. z- upper semicontinuous) in case $L_r(f)$ (resp. $L^r(f)$) is a zero set for each real number $r \in R$. These functions have been considered by Stone [24] any by Blatter and Seever [3] . Obviously the classes of all z-lower and z- upper semicontinuous functions are examples of $L(X)$ and $U(X)$ in theorem 0 (cf. [14]).

If X is a set and $\tilde{P} \subset 2^X$ a collection of subsets of X, then $\tilde{P}$ is called a paving, and the pair $(X, \tilde{P})$ a paved space, if $\tilde{P}$ is closed under finite intersections and countable unions and if X and $\emptyset$ belong to $\tilde{P}$. If $(X, \tilde{P})$ is a paved space, Y a topological space and $F : X \to Y$ a multifunction (i.e. a function, whose values are non-void subsets of Y), then F is called lower $\tilde{P}$-measurable, iff:

$$(2) \qquad F^{-}(G) := \{x \in X : F(x) \cap G \neq \emptyset\} \in \tilde{P}$$

holds for all open subsets $G \subset Y$. As an important example we have:

LEMMA 0. The family

$$(3) \quad \tilde{P}(X) := \{U \subset X : U = \{x \in X : f(x) > 0\} \text{ for some } f \in C^*(X)\} = \\ = \{X \setminus Z : Z = \{x \in X : f(x) = 0\} \text{ for some } f \in C^*(X)\}$$

consisting of cozero sets of $C^*(X)$ create a paving.

The simple proof will be ommited here. For related topics see [19], [20], [21].

A multifunction $F : X \to Y$, lower $\tilde{P}(X)$ - measurable with respect to (3) will be called z-lower semicontinuous (briefly z-lsc).

If card $F(x) = 1$ for all $x \in X$, i.e. $F(x) = \{f(x)\}$, then F is z-lsc if and only if f is continuous on X as a single-valued function.

A pair $(C, \tilde{S})$ is called a geometric complex, if C is a subset of a linear space and S is a covering of C by finite-dimensional simplices contained in C, such that $S \in \tilde{S}$ implies that all faces of S belong to $\tilde{S}$ and $S, T \in \tilde{S}$ implies that $S \cap T$ is a face of both S and T (or empty).

Let $(C,\tilde{S})$ be a complex. We denote by $V(C,\tilde{S})$ its set of vertices (i.e. the set of $x \in C$ such that $\{x\}$ belongs $\tilde{S}$) and call

$$\dim (C, \tilde{S}) := \sup \{\dim S : S \in \tilde{S}\} \text{ its dimension.}$$

For $y \in C$ let $S(y)$ be the simplex of smallest dimension in $\tilde{S}$, that contains y, and for $x \in V(C, \tilde{S})$ we call

(4) $\quad St (x) := \bigcup \{y \in C : x \in S(y)\}$

the star of x. C is always assumed to be topologized by the finest topology inducing the Euclidean topology on each $S \in \tilde{S}$. This topology is usually called the Whitehead topology.

If $\tilde{U}$ is a covering of a set X, a complex $N(\tilde{U})$, called its geometric nerve is assigned to $\tilde{U}$ in the following way: For $U \in \tilde{U}$ let $e_U : \tilde{U} \to R$ be defined by $e_U(V) = 0$ for $V \neq U$ and $e_U(U) = 1$. Let $\tilde{S}(\tilde{U}) := \{\text{conv} \{e_V : V \in \tilde{U}_1\} : \tilde{U}_1 \subset \tilde{U}$, $\tilde{U}_1$ is finite, $\bigcap \tilde{U}_1 \neq \emptyset\}$, $C(\tilde{U}) := \bigcup \tilde{S}(\tilde{U})$ and set $N(\tilde{U}) := (C(\tilde{U}), \tilde{S}(\tilde{U}))$.

For the remainder of the paper $\hat{k}$ is a cardinal number and α a nonnegative integer or ∞.

A paved space $(X, \tilde{P})$ is called $(\hat{k},\alpha)$- paracompact, iff every covering $\tilde{B} \subset \tilde{P}$ of X with card $\tilde{B} < \hat{k}$ admits a refinement $\tilde{B}^* \subset \tilde{P}$ such that :

(a) $\quad \dim N(\tilde{B}^*) \leq \alpha$

(b) $\quad$ there exists an $\tilde{P}$ - measurable map $\phi : X \to N(\tilde{B}^*)$ with $\phi^{-1} (St\, e_B) \subset B$ for all $B \in \tilde{B}^*$.

LEMMA 1. The paving (3) of cozero sets in an arbitrary space X is $(\aleph_1, \infty)$ - paracompact.

Proof. We start with an open covering $\tilde{B}$ of X which is

at most countable : $\tilde{B} = \{U_i : i \in I\}$, card $I \leq \aleph_0$.
Each U_i is of the form $U_i := \{x \in X: f_i(x) > 0\}$ for some f_i belonging to $C^*(X)$.
Let us define:

$$(5) \quad f(x) := \sum_{i \in I} 2^{-i} f_i^*(x), \text{ where } f_i^*(x) := 2^{-1}\left[1 + \frac{f_i(x)}{1+|f_i(x)|}\right]$$

Since the series (5) defining f converges uniformly, it follows that $f \in C^*(X)$. For each $x_o \in X$ there is an $i = i(x_o) \in I$ such that $x_o \in U_i$ since $\tilde{B}$ is a covering of X. Therefore $f_i(x_o) > 0$ and consequently $f(x) > 0$ on the whole space X. Define :

$$(6) \quad V^r := \{x \in X : f(x) > r\} = X \setminus L_r(f) \in \tilde{P}(X)$$

and observe that our cozero set V^n is an countable union of zero sets:

$$(7) \quad V^r = \bigcup_{n=1}^{\infty} \{x \in X : f(x) \geqslant r + 2^{-n}\} .$$

If $r = k^{-1}$ then put $V^r =: V_k$, $D_k := L^r(f)$, $D_o = \emptyset$ and define:

$$(8) \quad U_{kj} := U_j \cap (V_{k+1} \setminus D_{k-1}) \text{ for } 1 \leq j \leq k, \quad k=1,2,\ldots ,$$

$V_{kj} = \emptyset$ for $j > k$. For each $x_o \in X$ let us select $k = k(x_o) :=$ $: = \min\{k_1 \in N : x \in D_{k_1}\}$. Thus we have:

$$(9) \quad k^{-1} \leq f(x_o) < (k-1)^{-1} .$$

Observe that (9) implies that $x_o \in \bigcup_{j \leq k} U_j$. If not, then

$$f(x_o) = \sum_{i=k+1}^{\infty} 2^{-i} f_i^*(x_o) \leq \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k} < k^{-1}$$

in contradiction with (9). Consequently there is an index $j \leq k = k(x_o)$ such that $x_o \in U_j$. At the same time x_o belongs to $D_k \setminus D_{k-1}$, hence

(10) $x_o \in U_j \cap (D_k \setminus D_{k-1}) \subset U_j \cap (V_{k+1} \setminus \acute{D}_{k-1}) \subset U_{kj}$.

The first inclusion in (10) follows from the inclusion :

(11) $V_k \subset D_k \subset V_{k+1}$.

Next, observe that for each $j \leq k$ by virtue of (8) and (11) we have the following inclusion :

(12) $U_{kj} \subset V_{k+1} \subset D_{k+1}$. Thus :

(13) $\operatorname{card} \{(n,i) : U_{kj} \cap U_{ni} \neq \emptyset\} \leq 1+2+3+\dots+ k+1 = 2^{-1}(k^2 + 3k + 2) < \aleph_o$, $n \geqslant k+2$.

Indeed, if $p \in U_{kj} \cap U_{ni}$ then $(k-1)^{-1} > f(p) \geqslant (k+1)^{-1}$ and $(n-1)^{-1} > f(p) \geqslant (n+1)^{-1}$ so that $(k+1)^{-1} \leq f(p) < (n-1)^{-1}$, which in tourn implies $n \geqslant k+2$ so that:

$\{(n,i) : U_{kj} \cap U_{ni} \neq \emptyset\} = \{(1,1), (2,1), (2,2), (3,1), (3,2), \dots, (k+1,1), (k+1,2), \dots, (k+1, k+1)\}$ giving (13).

Consequently $\tilde{B}_o = \{U_{kj} : (k,j) \in N \times N\} =: \{U_s : s \in S\}$ defined by (8) create a star-finite subcovering of $\tilde{B}$. Since (8) are cozero sets in X, there is a partition of unity $\{g_s : s \in S\}$ subordinated to $\tilde{B}_o$. Define $\phi : X \to N(\tilde{B}_o)$ by formula :

(14) $$\phi(x) := \sum_{s \in S} g_s(x) \cdot e_{U_s}.$$

This function (14) is continuous : each $x \in X$ has a neighbourhood on which all but at most finitely many g_s vanish, and since this neighbourhood is mapped into a finite-dimensional flat in $C(\tilde{B}_o)$ and the addition is continuous, so ϕ is continuous on that neighbourhood, from which its continuity on the whole space

X results. Since $\sum_{s \in S} g_s(x) = 1$, then $\phi(x)$ is in fact a point of the closed geometric simplex spanned by $\{e_{U_s} : g_s(x) \neq 0\}$. The inverse image of $St\ e_{U_s}$ consists of all $x \in X$ for which $g_s(y) \neq 0$ and because the support of g_{U_s} is in U_s, we have $\phi^{-1}(St\ e_{U_s}) \subset U_s$ as required in (b). The item (a) is obvious.

The following definitions serve to formulate suitable conditions on the target space Y of our multifunction F. A map $H : 2^Y \to 2^Y$ is called a hull-operator on Y if $A \subset H(A) = H^2(A)$, $H(A) \subset H(B)$ for $A \subset B \subset Y$ and $H(\{y\}) = \{y\}$ for $y \in Y$ holds. A hull-operator H on a topological space Y is called α-convex, if the following is true: For every complex $(C, \tilde{S})$ with $\dim (C, \tilde{S}) \leq \alpha$ and every map $\rho : V(C,\tilde{S}) \to Y$ there exists a continuous map $\tau : C \to Y$ such that

(15) $\tau(S) \subset H(\rho(\text{ext}\ S))$ for all simplices $S \in \tilde{S}$.

The sign ext S means here the set of all extreme points (vertices) of a subset S.

Let Y be a set, $d: Y \times Y \to R$ a pseudometric on Y and H a hull-operator on Y. The function d is called H-convex if for all $A \subset Y$ with $A = H(A)$ and all $\varepsilon > 0$ we have:

(16) $$\{y \in Y : \text{dist}\ (y, A) := \inf_{a \in A} d(y,a) < \varepsilon\} = H(\{y \in Y: \text{dist}\ (y,A) < \varepsilon\}).$$

It Y is a uniform space [17], a hull-operator H on Y is called compatible with the uniform structure, if the uniformity ef Y is generated by a family of H-convex pseudometrics. However, if H is a compatible hull-operator en a metric space

(Y,d), the distance function d need not be H-convex.

A uniform space (Y, U) is called $\hat{k}$-bounded iff for any entourage $V \in \mathcal{U}$ of $(Y, \mathcal{U})$ there exists $Z \subset Y$ with card $Z < \hat{k}$, such that $Y = V(Z) := \{y \in Y : (z,y) \in V$ for some $z \in A\}$. If a uniform space contains a dense subset Z with card $Z < \hat{k}$, then it is obviously k-bounded. The following abstract selection theorem is proved in [17]:

THEOREM 1 ([17]). Let $(X, \tilde{P})$ be a $(\hat{k}, \alpha)$ - paracompact paved space, Y a $\hat{k}$-bounded complete metric space and H an α-convex, compatible hull-operator on Y. Then every lower $\tilde{P}$-measurable multifunction F between X and Y such that $F(x) = \mathrm{cl}\ F(x) = H(F(x))$ admits an $\tilde{P}$-measurable selector, i.e. a function $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$ and $L_r(f)$, $L^r(f) \in \{X \setminus G : G \in \tilde{P}\}$ for all $r \in R$.

Taking in the above theorem 1 the paving $\tilde{P}$ of the form (3), Y a separable Frechet space and $H(A) :=$ conv A, we obtain with the aid of our key lemma 1 the following :

PROPOSITION 1. Let X be an arbitrary topological space, Y a separable Frechet space and $F: X \to Y$ a z-lsc multifunction with closed, convex values. Then F admits a continuous selector $f: X \to Y$.

However theorem 0 may be treated as a particular case of proposition 1 since the multifunction $F(x) = [f_1(x), f_2(x)]$ is z-lsc if and only if both f_2 and $-f_1$ are z-lsc as the single-valued functions.

COROLLARY 1. Let (X,d,m) be a metric space with G_δ-regular, finite Borel measure without atoms such that

(17) $\inf \{m(K(x,r)) : x \in X, \ r > 0\} > 0,$

$K(x,r) := \{x_1 \in X : d(x_1,x) < r\}$

Let subsequently $(\tilde{F}, \Rightarrow)$ be a differentiation basis on X that means $\tilde{F} \subset 2^X$ is a preordered family of subsets $J \subset X$ with positive measure $m(J) > 0$ and $\Rightarrow$ is a convergence relation defined as follows :

(18) $(J_n) \Rightarrow x \Longleftrightarrow (\bigwedge_n (J_n \in \tilde{F} \wedge x \in J_n)) \wedge \lim_{n \to \infty} \operatorname{diam} J_n = 0.$

Suppose in addition that the following condition are fulfilled:

(A) $\bigwedge_{\varepsilon > 0} \bigwedge_{x \in X} \bigvee_{J \in \tilde{F}} (x \in J \wedge \operatorname{diam} J < \varepsilon)$

(B) $\bigvee_{L > 0} \bigwedge_{J \in \tilde{F}} m(\{x \in X : \operatorname{dist}(x,J) \leq 2 \operatorname{diam} J\}) \leq L \cdot m(J).$

(C) $\bigwedge_{A \in \tilde{M}} m(\{x \in A : \lim_{J_n \Rightarrow x} \frac{m(A \cap J_n)}{m(J_n)} < 1\}) = 0$

where $\tilde{M}$ is the m-completion of the borel tribe $\tilde{B}(X)$ of X. Let

(19) $\tilde{P}(X, T_d) := \{X \setminus A : A \in \tilde{M} \wedge D(A,x) = 1 \text{ for all } x \in A, \ A \in F_\sigma(X, d)\}$

where $D(A, x) := \lim_{J_n \Rightarrow x} [m(A \cap J_n) / m(J_n)].$

Then any lower $\tilde{P}(X, T_d)$ - measurable multifunction F defined on X and with closed, convex values in a separable Frechet space Y has an approximately continuous selector.

Proof: By virtue of the work of Chaika [5] our space (X, d, m) has the Lusin-Menchoff property from which we may easily deduce (cf. [8]) that $\tilde{P}(X, T_d)$ is a paving of exactly cozero sets of approximately continuous functions, i.e. functions belonging to $C(X, T_d)$, where

(20) $T_d := \{ G \subset X : D(G,x) = 1 \text{ for each } x \in G \}$

is so called density topology on X (cf. [15]). Then we may apply to the case under consideration the preceding proposition 1.

Let X,Y be as in the corollary 1. A multifunction $F: X \rightarrow Y$ will be called approximately z-lower semicontinuous if for every open subset $U \subset Y$ the set $F^-(U)$ belongs to $\tilde{P}(X,T_d)$ from (19) . We shall distinguish the z-lsc multifunctions from approximately lsc ones. Notice that approximately lsc multifunctions with compact, convex values may fail to have the approximately continuous selectores and may fail to be Borel- measurable, while approximately z-lsc multifunctions must belong to the lower Baire class 1. The notion of approximately continuous multifunctions were introduced and investigated by Hermes and lower approximately semicontinuous multifunctions appear in [25] . Note also that approximately z-lsc multifunctions with values being intervals on the real line appear in [27] under the name approximately lsc. In our opinion this name is in that context unadequate, since this notion is not the special case of the lower semicontinuity defined in [18], as the example 3 from [25] shows .

Following [27],[2] a multifunction $F : X \rightarrow Y$ has the property of approximate continuity on X if there exists an approximately z- lower semicontinuous multifunction $G : X \rightarrow Y$ with closed, convex values such that $G(x) \subset F(x)$ for every

$x \in X$ (i.e. G is a multiselector for an F).

COROLLARY 2. The multifunction $F: X \to Y$ admits an approximately continuous selector if an only if the has the property of approximate continuity on X.

Proof: The condition is obviously necessary, since we may take $G(x) = \{f(x)\}$ where f is the existing selector. The sufficiency comes from Corollary 1.

COROLLARY 3. If $A \subset X$ is a G_δ-subset of measure zero and $g : X \to Y$ is a Baire 1 vector-valued function, then there exists an approximately continuous function $f : X \to Y$ such that $g(x) = f(x)$ for every point x belonging to A.

Proof: Consider the multifunction defined as follows:

$$F(x) := \begin{cases} \{g(x)\} & \text{if } x \in A \\ \text{cl conv } g(A) & \text{otherwise .} \end{cases} \tag{21}$$

If G is open in Y, then $F^-(G) = X \setminus (A \setminus g^{-1}(G))$ if $G \cap$ cl conv $g(A) \neq \emptyset$ and $F^-(G)$ is empty whenever $G \cap$ cl conv $g(A) = \neq \emptyset$.

In both cases $F^-(G)$ is T_d - open and of the type F_σ. Thus F from (21) is z-lso on (X, T_d) and in compliance with corollary 1 has approximately continuous selector $f : X \to Y$. Obviously $f(x) = g(x)$ on A so that f is the desired extension of g.

From corollary 3 we directly obtain the following generalization of the prolongation theorem of Petruska and Laczkovich (cf. [23],[1],[7],[6],[10],[4]).

COROLLARY 4. (cf. [10] th. 2). Let $A \subset X$. The restriction to A of every Y-valued (bounded) Baire 1 function coincides with the restriction to A of a (bounded) approximately continuous function if and only if $m(A) = 0$.

Proof: If $m^*(A) > 0$, where m^* is the exterior measure on X generated by m, then there exists G_δ-superset $B \supset A$ such that $m(B) = m(A)$. Let $x_0 \in A$ be a point such that $D(B, x_0) = 1$. The function :

$$(22) \qquad g(x) := \begin{cases} y \neq 0 & \text{if} \quad x = x_0 \\ 0 & \text{if} \quad x \neq x_0 \end{cases}$$

is of the first Baire class, but each approximately continuous $f : X \to Y$ satisfy $A \not\subset \{x \in X : f(x) = g(x)\}$. If $m(A) = 0$ take G_δ-enveloppe $B \supset A$ with $m(B) = 0$ and then, applying corollary 3, we get a function $f: X \to Y$ such that $A \subset B \subset \{x \in X : f(x) = g(x)\}$ for an arbitrary Baire 1 function $g: X \to Y$. The proof is thereby completed.

Besides the topology T_d (20) we may consider in (X,d,m) another topology T_{ae} consisting of all subsets U of X for which :

(d) $U \in T_d$

(e) $U = G \cup Z$ where G is metricaly open and $m(Z) = 0$.

It is easy to observe that T_{ae} lies between the usual metrical topology and T_d and T_{ae} - continuous functions $C(X,T_{ae})$ are exactly those, which are approximately continuous every-where and metrically continuous m-almost everywhere (cf.[22], [11]). The following lemma characterizes the paving of cozero sets in (X, T_{ae}):

LEMMA 2. A function $f : X \to R$ is in $C^*(X, T_{ae})$

if and only if for each $r \in R$ we have :

(f) $\{x \in X : f(x) > r\} = X \setminus L_r(f) = G \cup Z$ where $G \cup Z$ is open in the density topology (20) , G is metrically open and Z is an F_σ - set of measure zero

(g) $L^r(f) := \{x \in X : f(x) \geq r\} = D \setminus Z$ where $D \setminus Z$ is closed in the density topology (20) , D is closed in (X, d) and Z is an F_σ set of measure zero.

Proof: We may assume that $G \cap Z = \emptyset$ (otherwise we may take $Z_o = Z \setminus G = Z \cap (X \setminus G) \in F_\sigma(X)$). Let us decompose Z onto the union $Z = \bigcup_{n=1}^{\infty} Z_n$ of closed sets $Z_n = \text{cl } Z_n$. By [5] (cf. also [15]) there is a perfect subset P_n such that $Z_n \subset P_n \subset G \cup Z_n$ and each point of Z_n is a point of density one for P_n . Next let us define f_n, $n = 1,2,\ldots$ as follows :

$$(23)\quad f_n(x) := \begin{cases} \dfrac{\text{dist}(x, X \setminus G)}{\text{dist}(x, X \setminus G) + \text{dist}(x, P_n)} & \text{if } x \notin Z_n \\ 1 \text{ if } x \in Z_n \end{cases}$$

where as in (16) and (B), dist (x, A) is the distance from the point x to the set A. It is easily seen that f_n from (23) is metrically continuous at each point $x \notin Z_n$ and is approximately continuous at each $z \in Z_n$. So, $f_n \in C(X, T_{ae})$. Also $X \setminus L_o(f_n) = G \cup Z_n$. Finally, put $X \ni x \longrightarrow f(x) := \sum_{n=1}^{\infty} 2^{-n} f_n(x) \in R$ as in formula (5) and observe that $f \in C^*(X, T_{ae})$ as well $\{x \in X : f(x) > 0\} = G \cup Z$. That achieves the proof of sufficiency. Necessity: Since f is T_d-continuous, we have $X \setminus L_r(f) \in T_d \cap F_\sigma(X)$. On the other hand, because of the metrical continuity m-almost everywhere of f, it follows from [16], th. 2a that $X \setminus L_r(f) = G \cup Z$ where G is open and Z is contained in an F_σ set of measure zero. Observe that $Z \setminus G = [X \setminus L_r(f)] \setminus G = \{x \in X : f(x) > r\} \cap (X \setminus G)$ is an F_σ set of measure zero. The proof is finished.

However it may be also easily observed, that the collection of cozero sets is a basis for the topology T_{ae}. Indeed, from lemma 2 we have :

$$(24) \quad P(X, T_{ae}) := \{G \cup Z : G \cup Z \in T_d, \ G \in T, \ Z \in F_\sigma(X,T), \ m(Z) = 0\} \subset T_{ae}$$

Let $U = G \cup Z \in T_{ae}$ and $x \in U$.
Then $G \cup \{x\} \in P(X, T_{ae})$. Clearly $x \in G \cup \{x\} \subset U$. Obviously $P(X, T_{ae})$ as a paving is closed under finite intersections and hence it create a basis for the topology T_{ae}.
Note, that this topology is completely regular, but not normal, similarly as in the case of T_d.

COROLLARY 5. Let (X,d,m) be a metric space with the distance function d and the measure m fulfilling all requirements of Corollary 1. Then any lower $P(X, T_{ae})$ - measurable multifunction $F : X \rightarrow Y$ defined on X and with closed, convex values in a separable Frechet space Y has an approximately continuous and m- almost everywhere metrically continuous selector.

Proof: By virtue of lemma 2, $P(X, T_{ae})$ is a paving of cozero sets of functions from the lattice $C^*(X, T_{ae})$ from which by using Proposition 1 we deduce our corollary.
In the sequel the spaces X, Y continue to be as in the

Corollary 1.

COROLLARY 6. If $Z \subset X$ is a closed subset with $m(Z)=0$ and $g:Z \to Y$ is an Baire 1 abstract function, then there exists an approximately and m - a.e. metrically continuous abstract function $f: X \to Y$ such that $g(x) = f(x)$ for every $x \in Z$.

Proof. Let us consider the multifunction $F: X \to Y$ given by the formula (21) from Corollary 3. Let G be open in Y such that $G \cap \mathrm{cl\ conv}\ g(Z) \neq \emptyset$. Observe that $m(Z \setminus g^{-1}(G)) = 0$ and $Z \setminus g^{-1}(G) = Z \cap (X \setminus g^{-1}(G)) \in G_\delta(X)$ so that $X \setminus (Z \setminus g^{-1}(G)) \in P(X, T_d)$. Moreover $X \setminus (Z \setminus g^{-1}(G)) = (X \setminus Z) \cup (g^{-1}(G) \cap Z)$.

Clearly $X \setminus Z$ is metrically open and $g^{-1}(G) \cap Z$ belongs to the $F_\sigma(X)$ and has measure zero. Thus $X \setminus (Z \setminus g^{-1}(G))$ belongs to the paving $P(X, T_{ae})$ of $C^*(X, T_{ae})$-cozero sets. The remaining case $F^-(G) = \emptyset$ is trivial. Thus F is z-lsc and in compliance with Corollary 5 has an approximately continuous and m - a.e. continuous selector $f: X \to Y$ coinciding with g on Z.

COROLLARY 7 (cf. [10], th. 3 p. 337) Let $A \subset X := R^n$. The restriction to A of every Y-valued (bounded) Baire 1 function coincides with the restriction to A of a (bounded) approximately continuous and m_n- a.e. metrically continuous function if and only if $m(\mathrm{cl}\ A) = 0$.

Proof: Necessity: Obviously $\mathrm{cl}\ A$ is always m-measurable. If $m(A) > 0$ then there is a subset $B \subset \mathrm{cl}\ A$ relatively nowhere dense, dense in itself and with the positive measure $m(B) > 0$. Let us select two disjoint countable subsets $A_1 \subset A$, $A_2 \subset A$ such that $B \subset \mathrm{cl}\ A_1 \cap \mathrm{cl}\ A_2$, $A_1 \cap A_2 = \emptyset$.

Take an arbitrary vector $y \in Y \setminus \{0\}$ an then put :

$$(25) \qquad A \ni x \longrightarrow g(x) := \begin{cases} y & \text{iff } x \in A_1 \\ 0 & \text{if } x \in A \setminus A_1 \end{cases}$$

It is easily checked that g belongs to the first Baire class on A, but each $f : X \longrightarrow Y$ with $f(x) = g(x)$ for all x belonging to A is totally discontinuous on B, viz. oso $f(x) = d_Y(y,0)$ at each $x \in B$. Thus f cannot be in $C^*(X, T_{ae})$. Sufficiency is a standard proof employing Corollary 6 for $Z := \mathrm{cl}\ A$.

Indeed, we obtain in such a manner a function $f : X \longrightarrow Y$ fulfilling $A \subset \mathrm{cl}\ A \subset \{x \in X : f(x) = g(x)\}$ for an arbitrary Y-valued function g belonging to the Baire 1 class on X.

REMARK. The space X in Corollary 7 may be endowed with an ordinary differentiation basis F consisting of those rectangles $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ for which the following inequality holds:

$$(26) \qquad K^{-1} < (b_i - a_i)/(b_j - a_j) < K \quad \text{for all } i \neq j \in \{1,2,\ldots,n\}$$

and some positive constant $K > 0$. The measure m_n may be the n-dimensional Lebesgue measure as well as a more general one fulfilling all requirements of Coroll. 1. Note that the same proof of necessity works in the more general case of certain ultrametric spaces instead of R^n, while in the sufficiency any additional assumption concerning the distance function d is clearly superfluous.

The Corollary 7 solves plainly the problem 13a from [9] and at the same time generalizes the theorem 3 from [10] in several directions. The subsequent proposition gives an negative answer to the next problem 13b from [9] :

PROPOSITION 3. There is a subset $A \subset R^2$ with $m_2(\mathrm{cl}\ A) = 0$ and a Baire 1 function $g : R^2 \to R$ such that for any $T_d \times T_d$ - continuous, m_2 - almost everywhere continuous function $f : R^2 \to R$, A is not contained in the set $\{(x,y) \in R^2 : f(x,y) = g(x,y)\} := (f-g)^{-1}(\{0\})$.

The sign m_2 denotes here the two-dimensional Lebesgue measure on the plane.

Proof: Let $A := \{5\} \times R$ and let us put :

$$(27) \qquad g(x,y) := \mathrm{sgn}\ y := \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y = 0 \\ -1 & \text{for } y < 0 \end{cases}$$

The function g from (27) is clearly Baire 1 and $m_2(A) = 0$. Let us suppose that $f : R^2 \to R$ is m_2 - a.e. continuous, $T_d \times T_d$ - approximately continuous function for which $A < \{(x,y) : f(x,y) = g(x,y)\}$. Observe that the following equality must holds : $f(5,y) = g(5,y) = \mathrm{sgn}\ y$ so that the section f_5 fails to have the Darboux property. Bearing in mind that any section of $T_d \times T_d$ - continuous function must be T_d-continuous and that all T_d-continuous functions are Darboux Baire 1 ones we obtain a contradiction. Thus $(f-g)^{-1}(\{0\})$ cannot be superset of A and the proof is completed.

The remaining question 13c from [9] is to prove or disprove the following Grande's conjecture: Let A be a subset of the plane R^2. The following sentences are the equivalent ;

(1^0) $m((\mathrm{cl}\ A)_x) = m((\mathrm{cl}\ A)^y) = 0$ where $(\mathrm{cl}\ A)_x = \{y \in R : (x,y) \in \mathrm{cl}\ A\}$ and $(\mathrm{cl}\ A)^y = \{x \in R : (x,y) \in \mathrm{cl}\ A\}$.

(2°) for each Baire 1 function $g: R^2 \to R$ there is m_2-a.e. continuous and approximately continuous with respect to the strong differentiation basis function $f: R^2 \to R$ such that $A \subset \{(x,y) \in R^2 : f(x,y) = g(x,y)\}$. Let us recall that a strong differentiation basis consists of all rectangles $[a_1, b_1] \times$ $\times [a_2, b_2]$ without no conditions (in the spirit of (26)) imposed upon the ratio $(b_2-a_2)/(b_1-a_1)$. Using the methods developed in this article, we may reduce that problem to finding of all cozero sets of strongly approximately continuous functions on the plane. In particular we have the following open questions:

Question 1. Let $A \in F_\sigma(R^2)$ be a subset such that :

$$(1^{oo}) \qquad D_2((x,y),A) := \lim_{\substack{h \to 0 \\ k \to 0}} \frac{m_2(A \cap ([x-h, x+h] \times [y-k, y+k]))}{4hk} = 1$$

$$(2^{oo}) \qquad \lim_{h \to 0} \frac{m_1(A^y \cap [x-h, x+h])}{2h} = 1$$

$$(3^{oo}) \qquad \lim_{k \to 0} \frac{m_1(A_x \cap [y-k, y+k])}{2k} = 1 \; ; \; x, y \in R .$$

Does there exist a function $f: R^2 \to R$ strongly approximately continuous such that $\{(x,y) : f(x,y) > 0\} = A$?

Question 2. Characterize the cozero sets for d_{xy}-continuous functions $f: R^2 \to R$ where d_{xy} is a topology recently introduced by O'Malley in the following way: a measurable subset $A \subset R^2$ is d_{xy} - open iff every x-section A_x and every y-section A^y are T_d-open, i.e. the condition (2^{oo}) and (3^{oo}) from Question 1 are fulfilled. A similar question can be raised for the topology q_{xy} consisting all subsets $A \subset R^2$ with the Baire property whose all sections A_x, A^y are qualitatively open, next for the topology q^o_{xy} consisting of all subsets $A \subset R^2$ with the Baire property and sections A_x, A^y metrically open and for the topology q^+_{xy} of all sets A with Baire property with all sections A_x, A^y I - continuous with respect to the Wilczyński category analogue of the density topology, etc. We have $q^{\bullet}_{xy} \subset q_{xy} \subset q^+_{xy}$ with proper inclusions.

A solution of each of those questions leads to some new prolongation theorems.

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O PRZEDŁUŻANIU FUNKCJI I KLASY BAIRE'A DO FUNKCJI A.E-CIĄGŁYCH

Streszczenie

W pracy wprowadza się pojęcie z-półciągłej z dołu multifunkcji, Pokazuje się następnie, że do takich multifunkcji stosuje się

twierdzenie Mägela o istnieniu selektora mierzalnego ze względu na paving zbiorów kozerowych kraty funkcji ciągłych określonych na dowolnej (nie koniecznie doskonale normalnej) przestrzeni topologicznej - taki selektor jest oczywiście ciągły. Uzyskane twierdzenie uogólnia dobrze znane wyniki Michaela. W dalszej części pracy stosujemy je do badania istnienia selektorów aproksymatywnie ciągłych i a.e. ciągłych dla z-lsc multifunkcji określonych na pewnych przestrzeniach metrycznych wyposażonych w miarę. Istnienie tych selektorów pozwala na rozstrzygnięcie problemu 13a, b opublikowanego przez Z. Grandego w [9] a dotyczącego istnienia a.e. - ciągłego przedłużenia funkcji 1 klasy Baire'a. Metoda zastosowana w [10] istotnie wykorzystuje fakt, że dziedzina jest prostą rzeczywistą, natomiast nasz Wniosek 7, stanowiący główny wynik niniejszego artykułu nie wymaga tego rodzaju ograniczeń. Dla kompletności w pracy należało przedstawić b. obszerny aparat pojęciowy związany z twierdzeniem Mägerla, pozwoliło to jednak sprowadzić dowód Stwierdzenia 1 do sprawdzenia 2 prostych lematów. Otrzymane wyniki stanowią zarazem przeniesienie rezultatów P. Vetro [27] na przypadek multifunkcji o wartościach w przestrzeniach nieskończenie wymiarowych.