ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY

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WŁODZIMIERZ A. ŚLĘZAK

WSP w Bydgoszczy

ON PREPONDERANTLY EQUICONTINUOUS COLLECTIONS OF TRANSFORMATIONS

The purpose of this article is to show that a problem 11 posed by Z. Grande in [10] has an affirmative answer, even in a more general setting than it is required in [10]. At the same time we give a solution of the question stated at the end of [13] and we prove some related theorems. In what follows (X, d_X), (Y, d_Y), (Z, d_Z) denote three separable, complete metric spaces, the first of which is equiped with a positive Borel measure m such that $m(K(x^o, r)) < +\infty$ and $\inf\{m(K(x^o,r)) : x^o \in X\} > 0$ for all $r > 0$ where $K(x^o, r) := \{x \in X : d_X(x^o, x) < r\}$ is an open ball centered at $x^o \in X$ and with radius r. R denotes as usually the real line endowed with the euclidean distance. Given an arbitrary set F we denote the space of all bounded transformations on F whose target space is Z by $B(F, Z)$. This space is completely metrized by the uniform metric D defined by:

$$D(h_1, h_2) := \sup \{ d_Z(h_1(f), h_2(f)) : f \in F\} .$$

By Z^X we denote the space of all transformations defined on X and with values in Z.

DEFINITION 1. A family F of m-measurable transformations $f:X \to Z$ is said to be preponderantly equicontinuous (cf. [10], p. 22) if there is a multifunction E from X into the hyperspace of nonempty m-measurable subsets of X and a positive real-valued function $\delta : X \to R_+$ such that for all $x^o \in X$ we have :

(a) $x^o \in E(x) \cap \operatorname{der} E(x^o)$, where der $E(x)$ denotes the set of all accumulation points of $E(x)$;

(b) the ratio $m(U(x^o) \cap E(x^o))/ m(U(x^o))$ is greather than 1/2 whenever $U(x^o)$ is an open neighbourhood of x^o whose diameter $\operatorname{diam} U(x^o) := \sup d_X(U(x^o) \times U(x^o)) < \delta(x)$

(c) the restrictions $\{f \mid E(x^o) : f \in F\}$ create a family equicontinuous at x^o. This means that :

(1) $$\bigwedge_{\varepsilon>0} \bigvee_{r>0} \bigwedge_{f \in F} \bigwedge_{x \in X} [x \quad E(x^o) \cap K(x^o, r) \Longrightarrow$$

$$\Longrightarrow d_Z(f(x^o), f(x)) < \varepsilon]$$

If $F = \{f\}$ consists of a single transformation $f:X \to Z$, then the above definition 1 reduces to preponderant continuity of f (cf. [5], [12], [18], [22]). Note that there is no topology T on X for which preponderantly continuous functions were exactly T- continuous, This follows from the fact that for two distinct preponderantly continuous at $x \in X$ functions $f,g : X \to R$ the measure $m(E^f(x) \cap E^g(x))$ may be arbitrarily small in each neighbourhood of x and thus f+g may fails to be preponderantly continuous at $x \in X$. In [13] Z. Grande has been introduced the following definition:

DEFINITION 2. A family $F \subset Z^X$ of transformations $f: X \to Z$ fulfils the property A_2 if for each nonvoid closed subset K of X there is a point $x^o \in K$ such that the restrictions $\{f|K : f \in F\} \subset Z^K$ form an equicontinuous at x^o collection of transformations ((1) with K instead of $E(x^o)$). We shall shortly writte $F \in A_2$ in that situation.

A more general notion has been investigated by Biagio Ricceri (Rocky Mountain J. of Math., vol. 14, no 3 (1984), pp. 503-517). Under his terminology the functions from the family $F \in A_2$ are equibelonging to the first Baire class. If F consists of a single transformation f, then $\{f\} \in A_2$ simply means that f is of the first Baire class [19] .

If $f : X \times Y \to Z$, we shall call a family of transformations $f_x : Y \to Z$, $x \in X$ defined by $f_x(y) := f(x,y)$, the X-sections of f. The Y-sections are defined similarly by $f^y(x) := f(x,y)$. Numerous papers were devoted to the conditions guaranteeing the Borel measurability of a transformation, expressed in terms of its sectionwise properties cf. a chart in [17], p. 169 . In particular [13] essentially contains the following deep theorem :

THEOREM 0. (cf. [13]). If $g : X \times Y \to Z$ is a transformation such that :

(d) $\{g^y : y \in Y\} \in A_2$ and

(e) all sections $g_x : Y \to Z$, $x \in X$ belong to the Baire class α, $0 < \alpha < \Omega$, then g also belongs to the Baire class α.

In case $X = Y = Z = R$ this is exactly the theorem 6 from [13] . Although the possibility of generalizing the domaine is not mentioned in Remark 3 on p. 125 in [13] , but this is evident by the penetrating inspection of the original proof. The generalization of the range space Z is permited, as it follows from the equality $g^{-1}(K(z,r)) = \{(x,y) \in X \times Y : d_Z(z, g(x,y)) < r\} \in F_\sigma(X \times Y)$ by virtue of the fact that for all $z \in Z$ a real-valued function $(x,y) \mapsto g_Z(x,y) := d_Z(z, g(x,y))$ fulfils assumptions (d) and (c) and each open set V in Z is a countable union of open balls in the presence of the separability of Z, cf. [15].

LEMMA 1. If $0 < d_X(x^1, x^2) < 3^{-1} \min(\delta(x^1), \delta(x^2))$ and

(2) $U(x^1, x^2) := \bigcap_{i=1}^{2} K(x^i, 3^{-1}\delta(x^i))$, then the intersection

(3) $E(x^1) \cap E(x^2) \cap U(x, x^2)$ is nonempty.

Proof: Observe, that $x^i \in U(x^1, x^2)$ for $i \in \{1,2\}$ and that

$$\operatorname{diam} U(x^1, x^2) \le \max_{1 \le i \le 2} \operatorname{diam} K(x^i, 3^{-1}\delta(x^i)) \le$$

$$\max_{1 \le i \le 2} 2 \cdot 3^{-1} \cdot \delta(x^i) < \max\{\delta(x^i) : 1 \le i \le 2\} .$$

Thus from the definition 1 we obtain the existence of numbers $r_i > 1/2$, $i \in \{1,2\}$ such that :

(4) $m(U(x^1, x^2) \cap E(x^i)) = r_i \cdot m(U(x^1, x^2))$, $i \in \{1,2\}$.

If the intersection (3) were empty, then

(5) $$m[U(x^1, x^2) \cap (E(x^1) \cup E(x^2))] = \sum_{i=1}^{2} m[E(x^i) \cap U(x^1, x^2)] =$$

$$= m(U(x^1, x^2)) \cdot \sum_{i=1}^{2} r_i > m(U(x^1, x^2))$$

in spite of the fact that (3) is a measurable subset of $U(x^1, x^2)$. Consequently these three sets must have a point in common, say $x^3 \in E(x^1) \cap E(x^2) \cap U(x^1, x^2)$, which proves our lemma.

DEFINITION 3 . (cf. [2]). Let $\delta : X \to R_+$ be a positive function and let K be a subset of X. By a δ-decomposition of K we shall mean a sequence of sets $\{K_n \subset K : n \in N\}$, which is a relabelling of the countable collection :

(6) $K^{mj} := \{x \in K : \delta(x) > 1/m\} \cap K(x_j^o, 2^{-1} m^{-1})$, where $\{x_j^o, j \in N\}$ is a countable dense set in X.

The key features of such a decomposition are recapitulated in a subsequent lemma:

LEMMA 2. Let $\{K_n : n \in N\}$ be a δ-decomposition of K. Then :

(i) $\bigcup_{n=1}^{\infty} K_n = K$

(ii) $x^1, x^2 \in K_n$ implies $d_X(x^1, x^2) < \min\{\delta(x^i) : 1 \leq i \leq 2\}$

(iii) if x_o belongs to the closure cl K_n of K_n then there are points $x \in K_n$ with $d_X(x_o, x) < 3^{-1} \min\{\delta(x_o), \delta(x)\}$.

Proof : If $x \in K$ then $\delta(x) > m^{-1}$ for some positive integer m and $d_X(x, x_j^o) < 2^{-1} m^{-1}$ for some $j = j(x,m) \in N$. Thus

$x \in K^{mj} =: K_n$ where $n = n(m, j) = 2^{m-1} \cdot (2j - 1)$ and (i) is proved. If $x^i \in K_n := K^{mj}$ then $\delta(x^i) > m^{-1}$ whenever $i \in \{1,2\}$.

By the triangle inequality we have:

$d_X(x^1, x^2) \leq d_X(x^1, x_j^o) + d_X(x_j^o, x^2) < 2^{-1} m^{-1} + 2^{-1} m^{-1} =$

$= m^{-1} < \min\{\delta(x^i) : i \in \{1,2\}\}$ and (ii) is proved.

If $x_o \in \mathrm{cl}\, K_n = \mathrm{cl}\, K^{mj}$ then there is a sequence $x^k \in K_n = K^{mj}$ convergent to x_o. Let $r \in (0, 4^{-1} m^{-1})$ be a number such that $d_X(x_o, x_j^o) = 2^{-1} m^{-1} - 2r$ and let $d_X(x^k, x_o) < r$ for all $k > k_o$.

Thus $d_X(x^k, x_j^o) \leq d_X(x^k, x_o) + d(x_o, x_j^o) = 2^{-1} m^{-1} - 2r + r = 2^{-1} m^{-1} - r < 2^{-1} m^{-1}$ and consequently $x^k \in K(x_j^o, 2^{-1} m^{-1})$ for $k > k_o$. Moreover we have $\delta(x^k) > m^{-1}$ and for sufficiently large $k > k_o$, $d_X(x^k, x_o) < 4^{-1} m^{-1} < 3^{-1} \min\{\delta(x^k), \delta(x_o)\}$ since x^k tends to x_o.

THEOREM 1. Each preponderantly equicontinuous family F of functions $f : X \to Z$ has the property A_2.

Proof: Without loss of generality we can suppose that the functions from the family F are uniformly bounded, i.e. there are a point $z \in Z$ and a positive number $M = M(z) > 0$ such that :

$$(7) \qquad \{f(x) \in Z \; : (x,f) \in X \times F\} \subset K(z, M) .$$

This follows from the fact that the formula (1) depends only on uniformity of the space Z and thus the particular distance functions may be replaced by the uniformly equivalent ones, e.g. $d := \min\{d_Z, 1\}$. Assume by a way of contradiction that F fails to have the A_2 property from definition 2 and yet is preponderantly equicontinuous in the meaning of definition 1. Then there exists a closed set $K_o \subset X$ such that :

$$(8) \qquad \bigwedge_{x_o \in K_o} \; \bigvee_{\varepsilon(x) > 0} \; \bigwedge_{\delta > 0} \; \bigvee_{x \in K_o} \; \bigvee_{f \in F} [d(f(x), f(x_o)) \geq \varepsilon \wedge d_X(x, x_o) < \delta] .$$

In other words

(9) $\bigwedge_{x_o \in K_o} \bigvee_{\varepsilon = \varepsilon(x_o) 0}$ osc $h(x_o) \geqslant \varepsilon$, where $h: X \to B(F, Z)$ is defined by the formula $h(x)(f) := f(x) \in (Z, d)$ and, as usually :

(10) $\operatorname{osc} h(x_o) := \inf \{ \sup \{ D(h(x), h(x_o)) : x \in K(x_o, \delta) \} : \delta > 0 \} =$

$= \inf \{ \sup \{ d(f(x), f(x_o)) : (x,f) \in K(x_o, \delta) \times F \} : \delta > 0 \}$.

We have $K_o = \bigcup_{n=1}^{\infty} K_n$, where for $n=1,2,...$ the set K_n is defined as follows :

(11) $K_n := \{ x \in K_o \subset X : \operatorname{osc} h(x) \geqslant n^{-1} \}$.

The function $(\operatorname{osc} h) : X \to R$ being upper semicontinuous, each of the sets (11) , $n \in N$, is closed in X. Since the set K_o is complete, as a closed subspace of a complete metric space X, then by famous Baire Category Theorem one of the sets K_n, $n \in N$ - by way of example K_m - is of the second category in X.

Let A_m denotes the relative interior of K_m in K_o and take $Q := \operatorname{cl} A_m$. We may assume that Q is a nonempty perfect set contained in X, with the property that the oscillation (10) of the restriction of h to Q exceeds m^{-1} at every point of Q. Let δ_1 be a positive function associated with multifunction E in definition 1 and choose a further positive function $\delta_2 : X \to R_+$ so that :

(12) $D(h(x^1), h(x^2)) < 1/6m$ for any x^2 belonging to $E(x^1)$ and satisfying $0 < d_X(x^1, x^2) < \delta_2(x^1)$. Let $\delta_3 := \min\{\delta_1, \delta_2\}$ and let $\{Q^n : n \in N\}$ be a δ_3-decomposition (see definition 3) of the set Q. By Baire's category

theorem invoked once again we can find that one of these subsets, say Q^k is dense somewhere :

$cl\,[Q^k \cap V] \supset V$ for certain subset V relatively open in Q. Let x^3, x^4 be any points in $Q \cap V$. By virtue of the density of Q^k and the item (iii) from Lemma 2 we may select the points x_k^3, x_k^4 belonging to Q^k so that :

(13) $d_X(x^i, x_k^i) < 3^{-1} \min\{\delta_3(x^i), \delta_3(x_k^i)\}$; $i \in \{3,4\}$. Define:

(14) $U(x, x_k^i) := K(x^i, 3^{-1}\delta_3(x^i)) \cap K(x_k^i, 3^{-1}\delta_3(x_k^i))$

for $i \in \{3, 4\}$ and observe that :

(15) $\operatorname{diam} U(x^i, x_k^i) \leq 2/3 \min\{\delta_3(x^i), \delta_3(x_k^i)\}$.

Then, by lemma 1, there are points $x_k^{i+2} \in E(x^i) \cap E(x_k^i) \cap U(x^i, x_k^i)$. From the definition (14) of $U(x^i, x_k^i)$ we have

(16) $$\max\{d_X(x_k^{i+2}, x^i), d_X(x_k^{i+2}, x_k^i)\} < \min\{\delta_3(x^i), \delta_3(x_k^i)\}.$$

Consequently :

(17) $$\max\{D(h(x_k^{i+2}), h(x_k^i)), D(h(x_k^{i+2}), h(x^i))\} < 1/6m.$$

On the other hand, from the item (ii) of lemma 2, we have :

(18) $d_X(x_k^3, x_k^4) < \min\{\delta_3(x_k^3), \delta_3(x_k^4)\}$.

Thus there exists a point $x_k^7 \in E(x_k^3) \cap E(x_k^4) \cap U(x_k^3, x_k^4)$ where $U(x_k^3, x_k^4)$ is defined by a similar manner as in (14). Therefore :

(19) $\max\{d_X(x_k^7, x_k^j) : j \in \{3, 4\}\} < \min\{\delta_3(x_k^j) : j \in \{3,4\}\}$

from which we obtain

(20) $D(h(x_k^7), h(x_k^j)) < 1/6m$ for $j \in \{3,4\}$.

Combing (17) and (20) together we obtain by the triangle inequality :

(21) $D(h(x^3), h(x^4)) \leq D(h(x^3), h(x_k^3)) + D(h(x_k^3), h(x_k^4)) +$

$+ D(h(x_k^4), h(x^4)) \leq D(h(x^3), h(x_k^5)) + D(h(x_k^5), h(x_k^3))$

$+ D(h(x_k^3), h(x_k^7)) + D(h(x_k^7), h(x_k^4)) + D(h(x_k^4), h(x_k^6))$

$+ D(h(x_k^6), h(x^4)) < 6 \cdot 1/6m = m^{-1}$.

But this contradicts our choice of $Q := \mathrm{cl}\, A_m$ and $m \in N$, since (21) means that $(\mathrm{osc}\, h)(x)$ cannot be greater than $1/m$ for $x \in Q$. Consequently (8) cannot be fulfilled and the family F must have the A_2 - property, as required. Hence the proof of our theorem 1 is completed.

Collating theorems 0 and 1 together we obtain:

COROLLARY 1. Let $g: X \times Y \to Z$ be a transformation whose all Y- sections $\{g(\cdot, y): y \in Y\} \subset Z^X$ create a preponderantly equicontinuous family and all X- sections $g_x := g(x, \cdot) \in Z^Y$, $x \in X$ belong to the Baire class α, $0 < \alpha < \Omega$. Then g belongs to the Baire class α too.

In my earlier paper [25] the transformations defined on the real line are investigated in a similar spirit. A notion of E- equicontinuity with respect to a system of path $E : X \to 2^X$ satisfying the intersection condition (cf. [2]) is introduced and a result similar to the above corollary 1 is obtained in such framework. In particular approximative equicontinuity (cf. [7]) and I- approximative equicontinuity (i.e. related to the category analogue of the density topology introduced by

Wilczyński, see [28]) is covered. However note, that the uniformity generated by the density topology (see [21])leads to the notion of approximative equicontinuity defined in [7] while the I- density topology of Wilczyński fails to be uniformizable. For the basis facts concerning uniform spaces the reader is refered to [23] .

Taking into account that the property A_2 implies in tourn the following property A_3 of the family $F \subset Z^X$:

$$(22)\quad F \in A_3 \iff \bigwedge_{x \in X} \bigwedge_{r>0} \bigvee_{x_0 \in K(x,r)} [F \text{ is equicontinuous at } x_0]$$

and modifying in a suitable manner the theorem 5 from [13] we are able to obtain from our theorem 1 the following:

COROLLARY 2. Let $g: X \times Y \to Z$ be a transformation whose all Y- sections create a preponderantly equicontinuous family and all X- sections are densely continuous (= cliquish). Then g is also densely continuous (= cliquish) as a transformation defined on the product space. Bearing in mind that we can allways replace d_Z by an uniformly equivalent bounded distance function d and slightly modifying the proof of theorem 7 from [13] we obtain immediately :

THEOREM 2. Any equi-upper semicontinuous family F of functions $f: X \to R$ has the property A_2. The same holds for equi-lower semicontinuity of F.

Let us recall (cf. [1],[4],[6],[9]) that a collection of functions $F \subset R^X$ is equi-upper semicontinuous at a point $x \in X$ if

$$(23)\quad \bigwedge_{\varepsilon>0} \bigvee_{\delta>0} \bigwedge_{f \in F} \bigwedge_{x \in X} [x \in K(x_0,\delta) \Rightarrow f(x) - f(x_0) < \varepsilon] .$$

The collection F is equi-upper semicontinuous if (23) holds for every $x \in X$. Equi-lower semicontinuity is defined in a similar manner or by replacing f by -f in the formula (23). At the present we are going to introduce a one-sided concept of preponderant equi-semicontinuity.

DEFINITION 4. A family F_1 of m-measurable real-valued functions $f: X \to R$ is said to be preponderantly upper semi-equicontinuous if there are a function $\delta: X \to R_+$ and a multifunction E exactly as in the definition 1 such that for all $x^o \in X$ conditions (a) and (b) from definition 1 are both satisfied and moreover

$$(24) \quad \bigwedge_{\varepsilon>0} \; \bigvee_{r>0} \; \bigwedge_{f \in F_1} \; \bigwedge_{x \in X} \left[x \in E(x^o) \cap K(x^o, r) \Longrightarrow \right.$$
$$\left. \Longrightarrow f(x) \in (-\infty, f(x^o) + \varepsilon) \right]$$

Sometimes the values of E are additionally demanded to be F_σ sets. A family $F_2 \subset R^X$ is called preponderantly lower semi-equicontinuous if $F_1 := \{-f : f \in F_2\}$ is prepondernatly upper semi-equicontinuous. If the above family F_i include a single function f_i, $i \in \{1,2\}$, then f is called upper (resp. lower) preponderantly semicontinuous. Notice, that there are preponderantly non-continuous functions, but simultaneously both lower and upper preponderantly semicontinuous (see an example in [12]). Let us suppose at present that our space Y is additionally endowed with a positive Borel measure m_Y satisfying a condition analogous to the condition imposed on m.
The subsequent theorem is an analogue of the th. 8, p. 20 from [7]:

THEOREM 3. Let $g: X \times Y \to R$ be a function whose all Y-sections are approximately ([14],[20]) upper semicontinuous and $F_2 := \{g(x,\cdot) : x \in X\} \subset R^Y$ is a preponderantly upper semiequicontinuous family. Then g is preponderantly upper semicontinuous on the product space $X \times Y$ endowed with the tensor product $m \otimes m_Y$ of measures.

Proof: Let $(x^o, y^o) \in X \times Y$ and let $\varepsilon > 0$ be given. There is a number $r_1 > 0$ such that

(25) $g(x, y^o) \in (-\infty, g(x^o, y^o) + \varepsilon/2)$ whenever $x \in E(x^o) \cap$
$\cap K(x^o, r_1)$ and $m[E(x^o) \cap U(x^o)] > (1-t)\, m\,(U(x^o))$ if
$\operatorname{diam} U(x^o) < \delta^2(x^o, t)$.

On the other hand, by the preponderant upper semiequicontinuity of the family F_2 we have :

(26) $g(x, y) \in (-\infty, g(x,y^o) + \varepsilon/2)$ whenever $x \in E(x^o) \cap$
$\cap K(x^o, r_1)$ and $y \in E(y^o) \cap K(y^o, r_2)$ for a suitable, sufficiently small $r_2 > 0$.

Define $E^2(x^o, y^o) := E\,(x^o) \times E^1(y^o)$. For all (x,y) belonging to the intersection $E^2(x^o, y^o) \cap K\,((x^o, y^o), r_3)$ where $r_3 := \min\{r_i : i \in \{1,2\}\}$ we have $g(x,y) - g(x^o, y^o) =$
$= g(x,y) - g(x,y^o) + g(x,y^o) - g(x^o, y^o) < \varepsilon/2 + \varepsilon/2 = \varepsilon$,
so that $g(x,y) \in (-\infty, g(x^o, y^o) + \varepsilon)$.

If $U^2(x^o, y^o)$ is contained in $U(x^o) \times V(y^o)$ then :

(27) $m_2\,[U^2(x^o, y^o) \cap E^2(x^o, y^o)] \geq$
$\geq m_2\,[(U(x^o)) \times V(y^o)) \cap (E(x^o) \times E^1(y^o))] =$
$= m_2\,[U(x^o) \cap E(x^o) \times V(y^o) \cap E^1(y^o)] = m\,[U(x^o) \cap$
$\cap E(x^o)]\, m_Y\,[V(y^o) \cap E^1(y^o)] = (1-t) \cdot m(U(x^o)) \cdot$
$\cdot (1/2 + t)\, m\,(V(y^o)) = [1/2 + t(1/2-t)]\, m_2(U(x^o) \times V(y^o)) >$

$> 1/2 \ m_2 \ U^2(x^o, y^o)$

whenever diam $V(y^o) < \delta^1(y^o)$. The sign m_2 means here $m \otimes m_Y$ and in X x Y the distance function $d_3((x^1, y^1), (x^2, y^2)) :=$ $\max\{d_X(x^1, x^2), d_Y(y^1, y^2)\}$ is selected.

Obviously a theorem similar to theorem 3 holds for functions with preponderantly lower semiequicontinuous sections (cf.[7], th. 9). The next theorem is in spirit of famous Kempisty's result [16].

We need the following lemma:

LEMMA 3. Suppose that a function $g : X \times Y \to R$ has all of its Y-sections preponderantly lower semicontinuous (not necessarily equisemicontinuous !). Then for each positive real constant s the function $g_s : X \times Y \to R$ define by the formula:

(28) $\quad g_s(x^o, y^o) := \sup\{g(x^o, y) : y \in K(y^o, s)\}$

is preponderantly lower semicontinuous on the product space $(X \times Y, d_3, m_2)$ where $m_2 := m \otimes m_Y$ and d_3 is defined at the end of the proof of theorem 3.

Proof: Let $(x^o, y^o) \in X \times Y$ be an arbitrary fixed point and let $\mathcal{E} > 0$ be given . By (28) there is a point y^1 belonging to the ball $K(y^o, s)$ such that $g(x^o, y^1) \in (g_s(x_o, y^o) - \mathcal{E}, +\infty)$. Since the section $g(\cdot, y^1)$ is preponderantly lower semicontinuous on X, there exists a radius $r_1 > 0$ such that for each $x \in E(x^o) \cap K(x^o, r_1)$ we have $g(x, y^1) \in (g_s(x^o, y^o) - \mathcal{E}, +\infty)$. Since $d_Y(y^o, y^1) < s$, there exists a number $r_2 > 0$ such that $d_Y(y^o, y^1) = s - r_2$. By the triangle inequality we have :

(29) $\quad d_Y(y^1, y) \leq d_Y(y^1, y^o) + d(y^o, y) < (s - r_2) + r_2 = s$

for each $y \in K(y^o, r_2)$. Thus y^1 belongs to the ball $K(y,s)$ whenever $y \in K(y^o, r_2)$. Consequently $g(x, y^1) \leq g_s(x,y)$ whenever $y \in K(y^o, r_2)$ and $x \in E(x^o) \cap K(x^o, r_1)$. But $g(x, y^1) \in \in (g_s(x^o, y^o) - \varepsilon, +\infty)$ so also $g_s(x,y) \in (g_s(x^o, y^o) - \varepsilon, +\infty)$ for all $(x,y) \in E(x^o) \cap K(x^o, r_1)] \times K(y^o, r_2) \supset E^2(x^o, y^o) \cap \cap K((x^o, y^o), r_3)$ where $E^2(x^o, y^o) := E(x^o) \times K(y^o, r^2)$, $r_3 := \min \{r_i, i \in \{1,2\}\}$ and $K((x^o, y^o), r_3) = K(x^o, r_3) \times K(y^o, r_3)$ grace a specific choise of a distance function d_2 on $X \times Y$. Observe that $(x^o, y^o) \in E^2(x^o, y^o) \wedge$ der $E^2(x^o, y^o)$ and that $m_2 [U(x^o) \times V(y^o) \cap E^2(x^o, y^o)] = m(U(x^o) \cap E(x^o)) \cdot m_Y(V(y^o) \cap K(y^o, r_2)) > 2^{-1} m(U(x^o)) m_Y(V(y^o)) = 2^{-1} m_2[U(x^o) \times V(y^o)]$ whenever $V(y^o) \subset K(y^o, r_2)$. Hence $m_2(U^2(x^o, y^o) \cap \cap E^2(x^o, y^o)) > 2^{-1} m_2(U^2(x^o, y^o))$ provided $\operatorname{diam} U^2(x^o,y^o) < < \delta^2(x^o, y^o) := \min \{\delta(x^o), r_2\}$ where δ is a function from the item (b) of def. 1. Since $(x^o, y^o) \in X \times Y$ was arbitrary, we have defined a multifunction $(x^o, y^o) \mapsto E^2(x^o, y^o)$ and two positive functions $(x^o, y^o) \mapsto \delta^2(x^o, y^o)$, $(x^o, y^o) \mapsto r(x^o,y^o) := r_3$ satisfying mutatis mutandis all requirements of definition 4 . Observe however that $E^2(x,y) \in F_\sigma(X \times Y)$ iff $E(x) \in \in F_\sigma(X)$. Thus $g_s : X \times Y \to R$ is preponderantly lower semicontinuous jointly, as a function of two variables.

THEOREM 4. Let $g : X \times Y \to R$ be a function whose all Y- sections are preponderantly lower semicontinuous and all Y- sections are d_Y- upper semicontinuous. Then g is a limit of a decreasing sequence of preponderantly lower semicontinuous functions.

Proof: Take an arbitrary sequence $s_1 > s_2 > \dots > 0$ tending decreasingly to zero and observe that because of the assumed d_Y - upper semicontinuity of Y- sections we have $g(x, y) = \lim_{s \to 0^+} g_s(x, y) = \lim_{n \to \infty} g_{s_n}(x, y)$ where g_s are defined by (28). Moreover for all $n \in N$ the following inequality:

(30) $\sup g_x^* K(y, s_{n+1}) =: g_{s_{n+1}}(x, y) \leq g_{s_n}(x, y) :=$
$\sup g_{s_n}^* K(y, s_n)$

holds, since $K(y, s_{n+1}) \subset K(y, s_n)$ for $y \in Y$. That observation achieves the proof. Under the continuum hypothesis one can construct a nonmeasurable function $g : X \times Y \to R$ with approximately lower semicontinuous X - sections and approximately upper semicontinuous Y- sections. Let us remark that paper [11] contains a theorem similar to our theorem 4 but concerning qualitative semicontinuity under the following rather artificial condition imposed upon d_Y :

(31) $$\bigwedge_{y_0} \cdot \bigwedge_{r>0} \bigwedge_{y_1 \in K(y_0,r)} \bigwedge_{y \in Y} [y \in K(y_0, \operatorname{dist}(y_1, \operatorname{Fr} K(y_0,r)))$$
$$\Rightarrow y_1 \in K(y,r)] .$$

An inspection of our proof shows that the condition (31) in [11] is superfluous. Our method also allows us to generalize onto the case of arbitrary metric spaces the theorem 6 from [7] in which the space Y is needlessly assumed to be euclidean and finite-dimensional. Finally, we give a theorem related to the results from [3] and [24] .

DEFINITION 5 (cf. [27]). A transformation $f: X \to Z$ is said to be non-alternating (in the sense of Whyburn) if, whenever C is connected in Z, its inverse image $f^{-1}(C)$ is connected in X.

Observe, that in the case where X=Z=R definition 5 reduces

to f being (weakly) increasing or decreasing.
In the sequel we shall assume additionally that the space Z has in addition the property, that each ball in Z is connected, and that $X = R$.

THEOREM 5. Let $f: X \times Y \to Z$ be a transformation whose all Y- sections are non-alternating and all X- sections create a separable subspace of the space $B_1(Y, Z)$ of Baire 1 bounded transformations. Then f is also of the first Baire class.

Proof: Let us put $h(x) := f_x \in B_1(Y, Z)$. We prove that h is a transformation of the first Baire class. Since the target space $h * X$ is separable, each open set in this image is a countable union of open balls. On the other hand each open ball $K(g,r)$ is a countable union of the closed balls $\bar{K}(g, r-2^{-n})$, $n=1,2,\ldots$. Therefore it suffices to prove that inverse images $h^{-1}(\bar{K}(g, r-2^{-n}))$ are subsets of X of the type F_σ.
Indeed, we have :

(32) $h^{-1}(\bar{K}(g,s)) = \{x \in X : D(h(x), g) \leq s\} = \{x \in X : d_1(f(x,y), g(y)) \leq s$ for each $y \in Y\} = \bigcap_{y \in Y} (f^y)^{-1}(\{z \in Z : d_1(z, g(y)) \leq s\})$.

All the balls $\bar{K}(g(y), s) \subset Z$ are connected on the strength of our additional assumption imposed upon the space Z. Bearing in mind, that the section f^y, $y \in Y$ are non-alternating, we conclude without difficulty that $(f^y)^{-1}(K(g(y), s))$ is connected and thus also convex, provided that X is the real line. Hence $h^{-1}(\bar{K}(g,s))$ is convex as the intersection of the indexed family of convex sets. Since each convex subset of the real line is ambiguous, therefore $h^{-1}(U) \in F_\sigma(X)$ for each open subset $U \subset h * X$ provided U is a countable union of closed balls. Consequently $h : X \to B_1(Y, Z)$ is of the first Baire class and has the separable range . Observe that $f(x,y) = h(x)(y)$ so that, by virtue of Baire theorem, the Y- sections of f fulfil the property A_2 . Invoking the theorem 0 with $\alpha = 1$ we obtain the claimed assertion. Note, that the space X may be generalized to be e.g. a curve in euclidean space, in particular a circle, i.e. a topological space without no order relation compatible with topology.

COROLLARY 3. Assume additionally that Y is compact metric space. Let $f : X \times Y \to Z$ be a transformation with non-alternating Y- sections and continuous X- section. Then f is in the first Baire class.

Proof: The space $C(Y, Z)$ endowed with the uniform metric

$$(33) \qquad D(g_1, g_2) := \sup \{d_1(g_1(y), g_2(y)) : y \in Y\}$$

is separable in the presence of compactness of Y and separability of Z. Thus we may apply the last theorem 5. In case where $Z = R = \bigcup_{k=-\infty}^{+\infty} [-k, k]$ this corollary gives a negative answer to the question 3 a,g from [10]. In connection with Corollary 3 let us recollect, that by an old result of H.D. Ursell [26] a function $f : R^2 \to R$ with isotonic Y- sections and L- measurable X- sections is L- measurable on the plane. Obviously this result may be generalized in a style of theorem 5. On the other hand a function $f: R^2 \to R$ with nondecreasing both X- sections and Y- sections may fails to be Borel measurable. Paper [24] contains an example of function defined on the plane not belonging to the first Baire class, whose all X- sections are right-continuous and increasing while all Y- sections are decreasing.

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O PRZEWYŻSZAJĄCO JEDNAKOWO CIĄGŁYCH RODZINACH PRZEKSZTAŁCEŃ

Streszczenie

W pracy tej pokazano, że przewyższająco jednakowo ciągła rodzina przekształceń mierzalnej przestrzeni metrycznej w ośrodkową przestrzeń metryczną posiada wprowadzoną przez Grandego własność A_2. Jako wniosek otrzymuje się pełne rozwiązanie problemu 11 opublikowanego w trzecim zeszycie Problemów Matematycznych [10] . Wprowadzono również pojęcie przewyższająco jednakowo półciągłej rodziny odwzorowań i udowodniono 2 proste fakty dotyczące tego pojęcia. Pracę kończy twierdzenie o przynależności do pierwszej klasy Baire'a pewnego odwzorowania określonego na przestrzeni produktowej i o wartościach w przestrzeni metrycznej, stanowiące uogólnienie wcześniejszego wyniku autora [24].