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CONCERNING CONTINUOUS SELECTORS FOR MULTIFUNCTIONS WITH NONCONVEX VALUES

There are already many papers devoted to the investigations of the conditions under which a multifunction whose values fails to be convex admits a continuous selector see [2]-[8], [11], [13], [16]-[17]. The present one is mainly consecrated to the existence of continuous selectors for nonconvex multivalued maps defined on product spaces. Firstly we give some preliminaries.

Let X be any topological space. If with each element x of X we associate a nonempty subset $F(x)$ of another topological space Y, we say that $F:X \rightarrow Y$ is a set- valued function (= multifunction) of X into Y.
If $B \subset Y$ and $F: X \rightarrow Y$ then we define

(1) $F^+(B) := \{x \in X : F(x) \subset B\}$,

$F^-(B) := \{x \in X : F(x) \cap B \neq \emptyset\} = X \setminus F^+(Y \setminus B)$

where F^+ and F^- are resp. the upper and lower inverses of F.
We employ the theory of semicontinuous set- valued functions and of topologies on hyperspaces of subsets of Y as developed in [18], [14], [3], [1] and [20]. If $F : X \rightarrow Y$ then F is called upper (resp. lower) semicontinuous on X if the set $F^-(A)$ is closed (resp. open) in X whenever A is closed (resp. open)

in Y. Equivalently, F is usc (resp. lsc) on X if the set $F^{+}(A)$ is open (resp. closed) in X where A is open (resp. closed) subset of Y.

If $A \subset Y$, then the closure of A will be denoted by $\bar{A}$. Let us consider the classes

(2) $\quad I(Y) := \{A \subset Y : A = \bar{A}$ and $A \neq \emptyset\}$, $K(Y) := \{A \in I(Y) : A$ is compact$\}$.

The collection $B(Y)$ of all classes of the form

(3) $\quad [O_1, O_2, \ldots, O_n] := \{A \in I(Y) : A \subset \bigcup_{i=1}^{n} O_i,\ A \cap O_i \neq \emptyset,\ i = 1, 2, \ldots, n\}$,

with $O_1, O_2, \ldots, O_n$ all open in Y is a base for a topology on $I(Y)$ called the Vietoris or exponential topology. A subbase for this topology on $I(Y)$ is the collection $S(Y)$ consisting of all classes having one of the following forms :

(4) $\quad O^{+} := \{A \in I(Y) : A \subset O\}$, $O^{-} := \{A \in I(Y) : A \cap O \neq \emptyset\}$, with O open in Y. If $B \in B(Y)$, then by (3) and (4) we have :

(5) $\quad B = [O_1, O_2, \ldots, O_n] = O^{+} \cap \left(\bigcap_{i=1}^{n} O_i^{-}\right)$, where $O = \bigcup_{i=1}^{n} O_i$.

Henceforth, $K(Y)$ will be treated as a subspace of $I(Y)$, the underlying topology being the one defined above.

A multifunction $F: X \to I(Y)$ is called continuous if for each open hyperset G in $I(Y)$ the counterimage $F^{-1}(G)$ is open in X. It is clear from the definitions that a multifunction $F: X \to I(Y)$ is continuous if and only if it is both upper and lower semicontinuous.

If the space Y is metrizable by the distance function $d: Y \times Y \to R$ then the hyperspace $I(Y)$ is metrizable by the generalized Hausdorff metric:

(6) $h(A, B) := \max \{ \sup \{ \text{dist}(a, B): a \in A \},$
$\sup \{ \text{dist}(b, A): b \in B \} \}, \quad A, B \in I(Y),$

where $\text{dist}(a, B) := \inf \{ d(a,b): b \in B \}$. Notice, that two equivalent metrics d_1 and d_2 on Y do not necessarily induce equivalent metrics (6) in the hyperspace of bounded, closed subsets of Y. In case of nonbounded closed subsets (6) in fact is only the generalized metric in the sense of C.K. Jung [22] , but then we may define h_1 = arc tg∘h for obtain a bona fide metric. On the hyperspace K(Y) the topology induced by the distance function (6) coincides with the Vietoris one, while the relationships between the Hausdorff continuity and the above defined Vietoris continuity for multifunctions with values in I(Y) are the following: Any multivalued function $F: X \rightarrow I(Y)$ continuous with respect to the generalized Hausdorff (6) is lsc (cf.[24], lemma 1.4), but may fails to be usc (see H. M. Ko [23], proposition 1 and Ex 1).

For $Q: [0, T] \rightarrow K(Y)$, where $[0, T]$ is a compact interval on the real line and Y is a metric space endowed with the distance function d define the variation of Q on the subinterval $[t-s, t]$, $s > 0$ as follows . Let P denotes a partition of $[t-s, t]$, i.e. a finite collection of points $t-s = t_0 < t_1 < \dots < t_{k+1} = t$ and let $\hat{P}$ denote the set of all such partitions. For the fixed partition $\tilde{P}$ define :

(7) $V^{t}_{t-s}(Q, \tilde{P}) := \sum_{n=1}^{k} h(Q(t_{n+1}), Q(t_n))$, $V_{t-s}(Q) :=$
$\sup \{ V^{t}_{t-s}(Q, \tilde{P}) : \tilde{P} \in \hat{P} \}$

If Q is Hausdorff continuous and has bounded variation (7) then $V_o^t(Q)$ is finite for all $t \in [0,T]$ and continuous as a function of the variable t (see [19] theorem 101 on p.581, the identical proof of continuity applies in case of $V_o^t(Q)$)
Now, we are prepared to generalize the target space in theorem 2 from [11], p. 540, originally stated for the multifunction with values in finite-dimensional Euclidean spaces.

THEOREM 1. Let $X := [0,T]$ be a compact interval and Y an arbitrary metric space and let $F: X \Rightarrow K(Y)$ maps this interval into the hyperspace of compact non-void subsets of Y continuously. Then:

a) If Q has a bounded variation (7) in X, then Q admits a continuous selector q, i.e. a continuous single-values map $q: X \rightarrow Y$ with the property

(8) $q(x) \in Q(x)$ for all $x \in X$, $q \in C(X, Y)$

b) If Q satisfies the Lipschitz condition of the form :

(9) $h(Q(x), Q(t)) \leq K \cdot |x - t|$, $x, t \in X$, $K > 0$

then Q admits a Lipschitz continuous selector q satisfying:

(10) $d(q(x), q(t)) \leq K |x - t|$

with the same Lipschitz constant K.

Proof: Let the image $Q(X) := \bigcup_{x \in X} Q(x) \subset Y$ be embeded isometrically into the Banach space $\tilde{Y} = C(Q(X), R)$ of continuous real functions on $Q(X)$ endowed with the uniform norm. This embedding is explicitly given by the formula:

$$Q(X) \ni z \mapsto d(\cdot, z) \in C(Q(X), R)$$

where d denotes the distance function on Y restricted to

$Q(X)$. Observe that by virtue of the assumed continuity of our multifunction Q and by the compactness of X the image $Q(X)$ is also compact in Y and thus $C(Q(X), R)$ is actually a separable Banach function space.

For each positive integer k let us consider a partition of X given by points: $0, Tk^{-1}, 2T k^{-1}, \ldots, (k-1) Tk^{-1}, T$. Choose an arbitrary point y_0^k belonging to $Q(0)$ and define $y_1^k \in Q(T k^{-1})$ to be the metric projection of y_0^k onto $Q(T k^{-1})$ i.e. $d(y_0^k, y_1^k) = \text{dist}\,(y_0^k, Q(T k^{-1}))$.

Next choose inductively the points $y_j^k \in Q(jT k^{-1})$ such that $d(y_{j-1}^k, y_j^k) = \text{dist}\,(y_{j-1}^k, Q(jT k^{-1}))$. Define $q^k \in C(X, \tilde{Y})$ as the polygonal arc joining the above selected points y_j^k, $j \in \{0,1,\ldots,k\}$, namely:

(11) $\quad q(x) = t\, y_{i+1}^k + (1-t) y_i^k$ where $iT k^{-1} \leq x \leq (i+1)T k^{-1}$

and t_x is defined as follows: (12) $t_x := \dfrac{x - iT k^{-1}}{T k^{-1}}$ for $x \in [iT k^{-1}, (i+1)T k^{-1}]$

Observe that for any $x \in X$ and any k, there exists an integer $j = j(k)$ such that $|x - jT k^{-1}| < T k^{-1}$. For x belonging to $[(j-1) T k^{-1}]$ we have:

(12) $$\text{dist}\,(q^k(x), Q(x)) \leq d(q^k(x), q^k(jT k^{-1})) + \text{dist}\,(q^k(j T k^{-1}), Q(x)) \leq h(Q((j-1)T k^{-1}), Q(jT k^{-1})) + h(Q(jT k^{-1}), Q(x))$$

where the last inequality follows from the fact that

$$d(y_{j-1}^k, y_j^k) \geqslant d(q^k(x), y_j^k) \quad \text{by virtue of (11) and}$$

since

(13) $h(Q((j-1)\,T\,k^{-1}\,,\ Q(jT\,k^{-1})) \geqslant \mathrm{dist}\,(y^k_{j-1},\ Q(jT\,k^{-1}))=$
$= d(y^k_{j-1},\ y^k_j\,)$

on the strength of our choice of points y^k_j .

Next observe that for $x,t \in X$ and any k, if j, i are integers such that

(14) $|x - jT\,k^{-1}| < T\,k^{-1}$, $|t\ \ iT\,k^{-1}| < T\,k^{-1}$

we have from the triangle inequality :

(15) $d(q^k(t),\ q^k(x)) \leqslant d(q^k(x),\ q^k(jT\,k^{-1})) + \sum_{n=j}^{i-1} d(q^k((n+1)\cdot$
$\cdot T\,k^{-1}),\ \ q^k(nTk^{-1})) + d(q^k(iT\,k^{-1}),\ q^k(t)) \leq$
$\leq h(Q(x),\ Q(jT\,k^{-1})) + \sum_{n=j}^{i-1} h(Q((n+1)T\,k^{-1}),$
$Q(nT\,k^{-1})) + h(Q(t)\ ,\ Q(iT\,k^{-1}))$

Now, to show part (a) of the theorem, we first demonstrate, that $\{q^k\ :\ k=1,2,\dots\} \subset C(X,\ \tilde{Y})$ create an equicontinuous family of functions. Given an arbitrary but fixed positive number $\varepsilon > 0$ we choose an integer k^*_ε sufficiently large so that for $k \geqslant k^*_\varepsilon$ the following implication holds:

(16) $|t - x| < T\ \ k^{*-1}_\varepsilon$ implies $h(Q(t),\ Q(x)) < \varepsilon/\,3.$

Next, since Q is of bounded variation (7) and $t \mapsto V^t_o(Q)$ is continuous as a function of $t \in X$, and hence (bearing in mind that the domaine X is compact) uniformly continuous on X, we can choose a positive number $\delta > 0$ such that $V^b_a\ (Q) < \varepsilon/\,3$ whenever $|a-b| < \delta$. Since $|\,jT/k - iT/k| \leq$ $\leq |\,x - t\,| + 2T\,k^{-1}$ if k is greather than $4T/\delta$ and if $|t - x| < \delta/2$ then we have $V^{iT/k}_{jT/k}\ (Q) < \varepsilon/\ 3$. Then from (15) we have $d(q^k(x),\ q^k(t)) < \varepsilon/3\ + \varepsilon/3 + \varepsilon/3 = \varepsilon$ whenever $k \geqslant \max(4T/\delta, k^*_\varepsilon)$ and $|x - t| < \delta/2$. Thus equicontinuity is shown. By virtue of the definition (11) for each k the image $q^k(X)$ is contained in convex hull conv $Q(X)$, where $Q(X) := \bigcup_{x \in X} Q(x) \subset \tilde{Y}$. Our multifunction Q being continuous with compact values is clearly upper semicontinuous.

It is well known that the image $Q(X)$ of compact set X under usc multifunction Q is also compact in the underlying Banach space $\tilde{Y} := C(Q(X), R)$ (see for example papers [25] or [26]). On the other hand the convex hull of compact subset $Q(X)$ in Banach space $\tilde{Y}$ is also compact in $\tilde{Y}$. Thus all functions $q^k \in C(X, \tilde{Y})$ have the same common range space conv $Q(X)$, and this range space is compact. (Therefore also complete and separable). We are now in a position to apply the generalized Arzela-Ascoli theorem, in compliance with which the set $\{q^k : k = 1,2,...\}$ is precompact in the Banach function space $C(X, \tilde{Y})$ endowed with the norm $|||q||| := \sup\{\|q(x)\| : x \in X\}$, where $\|\cdot\|$ is the norm in $\tilde{Y}$.

Thus the sequence $(q^k)_1^\infty$, has a Cauchy subsequence and by virtue of the inequality (12) and from the continuity of Q and closedness of its values this subsequence is convergent to the limit q such that $q(x) \in Q(x)$ for each $x \in X$. The space $C(X, Y)$ being complete, this selector q is continuous. The proof of the part (a) is already completed. To show part (b) we assume, without loss of generality that in (15):

(17) $\qquad x \leq jT\,k^{-1} < \ldots < iT\,k^{-1} \leq t \in X.$

Then utilizing the Lipschitz condition (9) for Q, the inequality (15) becomes:

(18) $$d(q^k(x), q^k(t)) \leq K\left[((jT k^{-1}) - x) + \sum_{n=j}^{i-1} ((n+1\, T k^{-1} - nT k^{-1}) + (t - iT k^{-1})\right] = K \cdot |t - x|.$$

Because the right-hand side of (18) is independent on k, we infer that the family $\{q^k : k=1,2,..\}$ is equicontinuous and similarly as in part (a) we can find a subsequence $(q^k)_1^\infty$ converging uniformly to, say, $q \in C(X, Y)$ and this function q satisfies (10) as it is evident passing to the limit in (18) where k runs over the domaine of considered subsequence. Again, from the inequality (12) and the fact that each set $Q(x)$ is closed in Y (and also in $\tilde{Y}$), we conclude that $q(x) \in Q(x)$, $x \in X$ and thus q is the desired selector. The proof is achieved. The domaine in the above theorem 1 cannot be essentially generalized, as the following example, adapted from [12] shows :

THOREM 2. There is a continuous multifunction $F: R^3 \rightarrow\!\!\!\!\rightarrow K(R^3)$ satisfying the Lipschitz condition (9) but without any continuous selector .

Proof: The proof will be only outlined. It is based on the example 2 , p. 190 from [12] with inessential changes. Let us consider the polar coordinates in $X = R^3$, namely (r, θ, φ). For $r > 0$ let $F(r,\theta,\varphi)$ be the circle S^1 of radius R=r which lies in a plane passing through the origin parallel to the tangent plane to the sphere of radius r centered at the origin, such that (r, θ, φ) is the point of tangency. For r=0 let $F(0, \theta, \varphi) = \{(0,0,0)\}$. By direct computation one may verify that F is Lipschitzian and the number $K=2$ serves as a Lipschitz constant. Suppose F were to admit a continuous selector $f: R^3 \to R^3$. Then the restriction $f \mid S^2$ of this selector to the sphere $\{(r, \theta, \varphi): r=1\} \approx S^2$ would be a cross - section of the circle bundle over that sphere S^2. This contradicts the fact that the sphere S^2 does not admit a continuous unit tangent vector field. In fact, the existence of such a field

(19) $\langle f(1, \theta, \varphi) \mid e \rangle = 0$, $e = (1, \theta, \varphi) \in S^2$

is equivalent to the existence of the homotopy between the antipodal map $a(1, \theta, \varphi) = (1, -\theta, -\varphi)$ and the identity map. This homotopy in S^2 from 1 to a is given by the formula:

(20) $H(x,t) = (1 - 2t)x + 2\,\mathrm{sqrt}(t - t^2) f(x)$, $\|f(x)\| = 1$,
$x = (1, \theta, \varphi) \in S^2$

But the existence of (20) is clearly inconsistent with the fact that $\deg a = -1$. By the same reasoning, our multifunction does not admit a continuous selector in any neighbourhood of the origin. Thus the Lipschitz condition is not even sufficient for the existence of a local continuous selector. In general one cannot expect more than a Baire classe one selector.

QUESTION 1. Let $F: R^2 \to K(R^2)$ be a compact- valued multifunction satisfying the condition (9). Does F must have a continuous selector ? The next example explains that the condition that Q in theorem 1 has a bounded variation cannot be relaxed, even if the target space is finite-dimensional.

THEOREM 3. Let $Y = R^2$ and let X be the same as in the theorem 1. There is a continuous multifunction $Q: X \to K(Y)$ which does not admit a continuous selector (cf. [12], p. 189).

Proof: For $t \in [0,1]$ let us put:

(21) $S(t) := \{(x,y) \in R^2 : x = \cos s,\ y = \sin s : t \leq s \leq 2\pi\}$

and for $t > 0$ let us define the function $\tilde{A}: X \to L(Y, Y)$ by the formula:

(22) $$\hat{A}(t) := \begin{bmatrix} \cos t^{-1} & \sin t^{-1} \\ -\sin t^{-1} & \cos t^{-1} \end{bmatrix}$$

Then define

(23) $$X \ni t \to Q(t) := \begin{cases} \hat{A}(t)S(t) := \{\hat{A}(t)[x,y]^T : (x,y) \in S(t)\} \text{ if } t > 0, \\ S(0) \text{ if } t = 0. \end{cases}$$

To show that Q is continuous in the Hausdorff topology, it suffices to show that Q is continuous at $0 \in X$. But $h(Q(t), Q(0)) \leq \pi t$. Furthermore Q does not admit a continuous selector since the graph of Q

(24) $\mathrm{Gr}\, Q := \{(t,x,y) \in X \times Y : (x,y) \in Q(t)\}$

considered as a subset of the cylinder $S^1 \times [0,1]$ is not arcwise connected. In fact, the gap in $Q(t)$ for $t > 0$ will disconnect any arc $\{(t, q(t)) : t \in X\}$. Thus Q is continuous on X but there does not exist a continuous point-valued function $q: X \to Y$ with values $q(t) \in Q(t)$, $t \in X$.

We are going to unite the theorem 1 with the following result of Hasumi [10] (see also [9] for a simpler proof)

THEOREM 4 ([10]). Let X be an extremally disconnected topological space, Y a regular (T_3) Hausdorff space and $F: X \to Y$ a compact-valued upper semicontinuous multifunction from X into Y. Then there exists a continuous selector f for F.

Though Hasumi [10] assumed X to the Hausdorff, his proof works unchanged in the situation of the above statement and will, therefore, be omitted. Let us recall, that a topological space X is extremally disconnected if the closure of every open set in X is open.

REMARK 0. A penetrating inspection of the existing proofs of theorem 4 (see [9], [10]) also permits us to relaxe the assumptions about the regularity of multifunction F. Instead of to be usc it may be allowed to be pseudo upper semicontinuous only in case where Y is a metric space in which the closures of open balls are closed balls (for example any linear metric space has this property). A multifunction $F : X \to Y$ is called pseudo upper semicontinuous if the big invers images of closed balls $F^{-}(K(y,r))$, $y \in Y$ are all closed in X. However the notion of pseudo upper semicontinuity appears sometimes in the literature under different names. Observe, that in the proof of theorem 1 in [9], p. 5 we may take as A the field generated by open balls and its closures and then the proof of theorem 4 on p. 7 in [9] remains correct, so that the theorem carries over the present situation.

THEOREM 5. Let X be the same as in theorem1 , Z an extremally disconnected topological space, and Y a complete metric space fulfilling the assumptions from Remark 0 and let $F: X \times Z \to Y$ be a multifunction such that all Z- sections

(25) $X \ni z \to F^{z}(x) := F(x, z)$, $z \in Z$

fulfil the Lipschitz condition of the form

(26) $h(F(x, z), F(t, z)) \leq K(z) \cdot |x - t|$

and all X- sections $F_x : Z \multimap Y$ are upper semicontinuous. Assume that the values of F are compact subsets of Y. Under these hypotheses there is a continuous selector $f: X \times Z \to Y$ with $K(z)$ - Lipschitzian Z- sections.

Proof: Let $C(X, Y)$ denote the space of all continuous maps from X into Y. The so-called compact- open topology in $C(X, Y)$ is that having as sub-basis all sets $\{f \in C(X,Y): f(K) \subset G\}$ where $K \in K(X)$ is compact and $G \subset Y$ is open. Let $C_K(X, Y)$ be the subset of $C(X, Y)$ having Lipschitz constant K. Define the multifunction $P: Z \multimap C(X, Y)$ by formula:

(27) $$Z \ni z \mapsto P(z) := \{f \in C_{K(z)}(X, Y): f(x) \in F(x) \text{ for each } x \in X\}.$$

It is easy to observe, that the values of P are closed subsets of $C_{K(z)}(X, Y)$. We prove that (27) is pseudo upper semicontinuous multifunction from Z into $K(C(X,Y))$. In fact it suffices to prove, that $P^-(D)$ is closed in Z whenever D is a closed set of the form

$$D := \{f \in C(X,Y): \tilde{D}(f, f_o) := \sup \{d(f(x), f_o(x)) : x \in X\} \leq r\}$$

where $f_o \in C(X, Y)$ and $r > 0$ is a positive real number. Denote by $\bar{K}(f_o(x),r)$ the closed ball in Y centered at $f_o(x) \in Y$ and of radius r. We have:

(28) $$P^-(D) = \{z \in Z: P(z) \cap D \neq \emptyset\} = \{z \in Z: \bar{K}(f_o(x), r) \cap F(x,z) \neq \emptyset \text{ for all } x \in X\}.$$

Since all X- sections of F are usc, i.e. $F_x^-(\bar{K})$ is closed for each closed subset $\bar{K} \subset Y$, we infer that the set

(29) $$\{z \in Z : F(x, z) \cap \bar{K}(f_o(x), r) \neq \emptyset\}$$

is olosed in Z for each $x \in X$. Therefore :

$$(30) \quad P^{-}(D) = \bigcap_{x \in X} \{z : F(x,z) \cap \bar{K}(f_0(x), r) \neq \emptyset\} =$$

$$= \bigcap_{x \in X} F_x^{-} (\bar{K} (f_0(x), r)$$

is closed in Z for each x belonging to X. Consequently P is upper semicontinuous. By virtue of theorem 1 the values of (27) are nonvoid. The compaotness of the values of multifunction P follows by using the Arzela-Asooli theorem from the equicontinuity of the family $C_K(X,Y)$ and from the fact that for each $x \in X$ the seotion $\{f(x): f \in P(z)\}$ is oompact in Y by virtue of the point-compactness of our starting multifunction F. Applying theorem 4 we get the continuous function $p \in C(Z, C(X, Y))$ such that $p(z) \in P(z)$ for each $z \in Z$. Define $f(x, z) := p(z)(x)$. By standard argument $f \in C(X \times Z, Y)$. Moreover $f(x, z) \in F(x, z)$ for each $(x,z) \in X \times Z$ and F^z fulfills (10) with the Lipschitz constant $K(z)$. The proof (similar to the proof of theorem 6 from my previous paper [21]) is thereby achieved.

Let us remark, that the space X in theorem 5 may be assumed to be the whole real line $R = \bigcup_{n=-\infty}^{+\infty} [n, n+1]$, since the selectors q_n of $Q_n := F^z | [n, n+1]$ may be find with the additional property that $q_n(n+1) = q_{n+1}(n+1)$ and thus may be sticked to obtain desired seleotor.

QUESTION 2. Does there exists a multifunction $F: R \to R$ with closed (noncompact and nonconvex) values fulfilling (9) but posessing no continuous (lipsohitzian) seleotor ? The sign h in (9) in the present situation stands for generalized Hausdorff metric (in the meaning of Yung [22]).

Following [6] a metric space (X, d) is strongly connected if there exists a sequence $C_n \in K(X)$ of compact subsets such that :

a) $C_{n+1} \supset C_n$ for $n \in N$, $\bigcup_{n=1}^{\infty} C_n = X$

b) $\bigwedge_{x \in X} \bigvee_{n \in N} x \notin \operatorname{Int} C_n$

c) if K_o is an open ball in C_n and $t, x \in K_o$ are such that $d(x, t) = 2R$ then for each $r > 0$ the intersection $K_o \cap K(x, R+r) \cap K(t, R+n) \cap C_n \neq \emptyset$ is nonempty whenever $n \in N$. Denote by x_o certain point from this intersection.

d) if x_i, x_j belongs to C_n, $i, j \in \{0,1,2\}$ are such that $d(x_i, x_j) < 2 \cdot r_o$ then there exist points $t_k \in K(x_o, r_o) \cap \cap K(x_k, r_o) \cap C_n$, $k \in \{1,2\}$ such that $d(t_1, t_2) < r_o$ for all $n \in N$.

Each strongly connected metric space is connected, separable and locally compact, but it may be incomplete. Each convex compact subset of Banach space is strongly connected and each finite dimensional Hilbert space is strongly connected.

THEOREM 6. (cf. [5],[6]) . Let X be a strongly connected metric space and Y an arbitrary metric space. If $Q: X \to Y$ is a multifunction such that : 1^o card $Q(x) = n$ for each $x \in X$, 2^o Q is lower semicontinuous (resp. upper semicontinuous) then Q admits exactly n distinct continuous selectors:

(31) $Q(x) = \{q_1(x), q_2(x), \ldots, q_n(x)\}$, $q_i \in C(X, Y)$

Moreover $Q \mid C_n$ is continuous.

Combining theorems 6 and 4 we are able to prove the following:

THEOREM 7. Let X be strongly connected metric space, Z an extremally disconnected topological space and Y an arbitrary metric space. Consider a multifunction $F: X \times Z \to Y$ such that

(32) $\quad \text{card } F(x,z) = n$ for each $(x,z) \in X \times Z$

and such that all Z-section (25) are lower semicontinuous and all X- sections are upper semicontinuous. Then F has a continuous selector $f: X \times Z \to Y$.

Proof: Let us redefine D in the proof of theorem 5 as follows:

(33) $$D := \{f \in C(X, Y): \sup\{d(f(x), f_o(x)): x \in C_n\} \leq r\} = D(f_o, r, n)$$

and use the theorem 6 instead of theorem 1 to check that $P(z) \neq \emptyset$ for all $z \in Z$. The values of P are all of the cardinality $n \in N$ and thus are compact so that the theorem 4 can be invoked to obtain p. The desired selector f is defined similarly as in the proof of theorem [5].

If X in theorem 6 is normable and separable rather than strongly connected, and the space Y is Banach then if Q fulfills the Lipschitz condition (9), its selectors q_1 from (31) are also Lipschitzian with the same constant K. It would be interesting to known can be the above condition imposed on the space X relaxed. The possibility to obtain continuous selectors for multifunctions, whose Z-sections fulfills the condition of others existing theorems e.g. [8], [13] for multifunction with contractible values, [16] for multifunctions with starsheaped values will be investigated in a latter paper.

QUESTION 3. Does the target space Y in theorems 1 and 5 can be generalised to be an object of the category of Lipschitz spaces and Lipschitz maps which is defined similarly as in Fraser [27] . For any set Y let S(Y) be the family of all nonnegative functions b defined on the square Y x Y which are symmetric and vanishing on the diagonal: $b(y_1,y_2)= = b(y_2, y_1)$ and $b(y,y) = 0$ $y_1, y_2, y \in Y$.

A Lipschitz structure on Y is subset $L \subset S(Y)$ such that:

(L1) $\bigwedge_{y_1,y_2 \in Y} \bigvee_{b \in L} \; y_1 \neq y_2 \implies b(y_1,y_2) \neq 0$,

(L2) $\bigwedge_{b,c \in L} \bigwedge_{a \in S(Y)} \; a \leq b + c \implies a \in L$,

(L3) $\bigwedge_{a \in L} \bigvee_{b \in L} \bigwedge_{p,q,e \in Y} \; b(p,q) + b(q,e) \geq a(p, e)$.

A Lipschitz space is a pair (Y, L(Y)), where L(Y) is a Lipschitz structure on Y. Let (X, L(X)), (Y, L(Y)) be two Lipschitz spaces.

For $T: X \to Y$ we define $T^{\S} : S(Y) \to S(X)$ by the formula:

(34) $(T^{\S} b)(x,e) := b(Tx, Te)$ for every $b \in S(Y)$: $x,e \in X$.

A singlevalued map $T: X \to Y$ is called a Lipschitz operator iff $T^{\S} L(Y) \subset L(X)$. It is clear that every Lipschitz space (Y, L(Y)) carries the uniform structure determined by the entourages:

(35) $\{(y,e) \in Y \times Y : b(y,e) < t\}$, $b \in L(Y)$, $t > 0$.

Every Lipschitz operator is a uniform morphizm.

Any function $g: L(Y) \to L(X)$ is called a Lipschitz rank.

Denote:

$$g^{\xi} = \{ (z,u) \in (X \times X) \times (Y \times Y) : \bigwedge_{a \in L(Y)} a(u) \leq g(a)(z) \} \tag{36}$$

It is evident that $T: X \to Y$ is a Lipschitz operator iff there exists a rank g such that

$$\mathrm{Gr}\, T \times T := \{ ((x_1, x_2), (y_1, y_2)) \in (X \times X) \times (Y \times Y) : y_i = T(x_i),\ i = 1,2 \} \subset g^{\xi} . \tag{37}$$

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W SPRAWIE CIĄGŁYCH SELEKTORÓW DLA MULTIFUKCJI Z NIEWYPUKŁYMI WARTOŚCIAMI

Streszczenie

Głównym wynikiem tego artykułu jest dowód istnienia ciągłego selektora dla multifunkcji $F: X \times Z \to Y$ o zwartych wartościach spełniającej warunek Lipschitza ze względu na pierwszą zmienną oraz półciągłej z góry ze względu na drugą zmienną z osobna, przy czym o przestrzeni Z zakłada się, że jest ekstremalnie niespójna, a o przestrzeni X, że jest zwartym przedziałem prostej rzeczywistej, podczas gdy wartości są podzbiorami dowolnej zupełnej przestrzeni metrycznej. Wymagało to uogólnienia twierdzenia Hermesa z [11]. Dla kompletności

przytoczono zaadaptowane z literatury przykłady, że pewne inne nasuwające się kierunki uogólnień skazane są na niepowodzenie. Opracowaną dla twierdzenia 5 metodą dowodu zastosowano następnie do przypadku multifunkcji, której wartości są zbiorami skończonymi tej samej mocy, przy czym przestrzeń X może być wtedy jedynie silnie spójna w sensie Carbone, a lipschitzowskość cięć zastąpiona zostaje półciągłością z dołu.