ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY

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ON THE URYSOGN INTEGRAL EQUATION IN BANACH SPACES

1. Introduction

Let $D = [0,d]$ be a compact interval in R and let E be a real Banach space. Consider the integral equation

(1) $$x(t) = g(t) + \lambda \int_D f(t,s,x(s))\,ds ,$$

where $(t,s,x) \to f(t,s,x)$ is a function from $D^2 \times E$ into E which is continuous in x for a.e. $t,s \in D$ and strongly measurable in (t,s) for each $x \in E$. In [8] the following theorem was proved :

THEOREM 1. If the function f is such that

(i) the integral operator F defined by

$$F(x)(t) = \int_D f(t,s,x(s))\,ds$$

continuously maps an Orlicz space $L_\varphi(D,E)$ into itself ;

(ii) $\beta(f(t,s,Z)) \le H(t,s)\beta(Z)$ for a.e. $t,s \in D$ and for every bounded subset Z of E, where H is an appropriate function and β denotes the measure of noncompactness,

then for any $g \in L_\varphi(D,E)$ there exists a positive number r such that for each $\lambda \in R^*$ with $|\lambda| < r$ the equation (1) has a solution $x \in L_\varphi(D,E)$.

The proof of Theorem 1 essentially used a theorem of

Heinz [3 ; Th. 2.1] on the measurability of the function $t \to v(t) = \beta(V(t))$, where V is a given countable set of strongly measurable functions from D into E and $V(t) = \{x(t): x \in V\}$. Let us remark that the proof of Heinz result is very long and complicated.

In this paper we shall show that Theorem 1 can be proved without using of Heinz theorem whenever we replace the assumption $|\lambda| < r$ by $|\lambda| < r/2$. For simplicity, we restrict our considerations only to the space $L^2(D,E)$.

2. Measures of noncompactnes

Denote by $L^p(D,E)$ $(p \geq 1)$ the space of all strongly measurable functions $u: D \to E$ with $\int_D \|u(t)\|^p dt < \infty$, provided with the norm $\|u\|_p = (\int_D \|u(t)\|^p dt)^{1/p}$.

Let us recall that the Hausdorff measure of noncompactness β_X in a Banach space X is defined by

$$\beta_X(Z) = \inf\{\varepsilon > 0 : Z \text{ admits a finite } \varepsilon\text{-net in } X\}$$

for any bounded subset Z of X. For properties of β_X see [1, 7]. For convenience we shall denote by β and β_1 the Hausdorff measures of noncompactness in E and $L^1(D,E)$, respectively.

For any set V of functions from D into E we define a function v by $v(t) = \beta(V(t))$ $(t \in D)$, where $V(t) = \{x(t) : x \in V\}$ (under the convention that $\beta(Z) = \infty$ if Z is unbounded). The following lemma plays an important role in our existence proof.

Lemma [6; Th. 1].

Assume that the space E is separable and V is a countable set of functions belonging to $L^1(D, E)$. If there

exists a function $\mu \in L^1(D,R)$ such that $\|x(t)\| \leq \mu(t)$ for all $x \in V$ and $t \in D$, then the corresponding function v is integrable and for any measurable subset T of D

$$\beta(\{\int_T x(t)dt : x \in V\}) \leq \int_T v(t)dt .$$

Moreover, if $\lim_{h \to 0} \sup_{x \in V} \int_D \|x(t+h) - x(t)\|dt = 0$, then

$$\beta_1(V) \leq \int_D v(t)dt .$$

3. The main result

Assume that

1° $g \in L^2(D,E)$;

2° there exist nonnegative functions $a \in L^2(D,R)$ and $K \in L^2(D^2, R)$ such that $\|f(t,s,x)\| \leq K(t,s)(a(s) + b\|x\|)$ for $t,s \in D$ and $x \in E$, where b is a positive number.

Let F be the mapping defined by

$$F(x)(t) = \int_D f(t,s,x(s))\, ds \qquad (x \in L^2(D,E),\ t \in D).$$

It is known (cf. [4]) that under the assumptions 1° and 2° F continuously maps $L^2(D,E)$ into itself. Assume, in addition, that for any $r > 0$

3° $\lim_{h \to 0} \sup_{\|x\|_2 \leq r} \int_D \|F(x)(t+h) - F(x)(t)\| dt = 0.$

Theorem 2. If f and g satisfy $1^\circ - 3^\circ$ and there exists a nonnegative function $H \in L^2(D^2, R)$ such that

$$(2) \qquad \beta(f(t,s,Z)) \leq H(t,s)\beta(Z)$$

for $t,s \in D$ and for each bounded subset Z of E, then there exists a positive number ρ such that for any $\lambda \in R$ with $|\lambda| < \rho$ the equation (1) has a solution $x \in L^2(D,E)$.

PROOF. Put

$$\varrho = \min \left(\sup_{r>0} \frac{r - \|g\|_2}{\|K\|_2 (\|a\|_2 + br)}, \frac{1}{4\|H\|_2} \right).$$

Fix $\lambda \in R$ with $|\lambda| < \varrho$. Then there exists $r > 0$ such that

$$\|g\|_2 + |\lambda| \, \|K\|_2 (\|a\|_2 + br) \leq r .$$

Let $B = \{x \in L^2(D,E) : \|x\|_2 \leq r\}$ and $G(x) = g + \lambda F(x)$ for $x \in B$. Then G is a continuous mapping $B \to B$ and

(3) $\|G(x)(t)\| \leq \mu(t)$ for $x \in B$ and $t \in D$,

where $\mu(t) = g(t) + |\lambda| \, \|K(t,\cdot)\|_2 (\|a\|_2 + br)$.

Let $V = \{u_n : u \in N\}$ be a countable subset of B such that

(4) $V \subset \overline{\operatorname{conv}} \, (G(V) \cup \{0\})$.

Then there exists a subset A of D such that $\operatorname{mes} (D \setminus A) = 0$ and

(5) $V(t) \subset \overline{\operatorname{conv} \, (G(V)(t) \cup \{0\}}$ for $t \in A$.

For any $m \in N$ we define an operator S_m by

$$S_m x \, (t) = \frac{m}{2} \int_{t-1/m}^{t+1/m} x(s)ds \qquad (x \in L^1(D,E), \; t \in D).$$

It is well known that for any $x \in L^1(D,E)$ the function $S_m x$ is continuous on D and $\lim_{m \to \infty} \|S_m x - x\|_1 = 0$. Hence for a given $x \in L^1(D,E)$ there exists a subset $D(x)$ of D such that $\operatorname{mes} (D \setminus D(x)) = 0$ and the sequence $(S_m x)$ has a subsequence which converges to x in all points $t \in D(x)$.

Let $P = A \cap D(g) \cap D(u_1) \cap D(u_2) \cap \ldots$. Denote by X the closure of linear hull of the set $\{S_m g(t_i), S_m u_n(t_i) : m,n,i \in N\}$, where (t_i) is a dense sequence in P.

Fix now $t \in P$ and put $y_n(s) = f(t,s,u_n(s))$ for $s \in D$.

Let $Q = P \cap D(y_1) \cap D(y_2) \cap \dots$ and let Y be the closure of linear hull of the set $\{S_m g(t_i), S_m u_n(t_i), S_m y_n(s_i): m, n, i \in N\}$, where (s_i) is a dense sequence in Q.
From the above construction it is clear that X and Y are separable Banach subspaces of E and

$$
(6) \qquad \begin{array}{ll} g(s),\ u_n(s) \in X & \text{for all } n \in N \text{ and } s \in P \\ y_n(s) \in Y & \text{for all } n \in N \text{ and } s \in Q\ . \end{array}
$$

Moreover, $X \subset Y$, mes $(D \setminus P) = 0$ and mes $(D \setminus Q) = 0$. On the other hand, owing to (3) and (4), we have

$$
(7) \qquad \|u_n(s)\| \le \mu(s) \quad \text{for a.e. } s \in D \text{ and } n \in N\ ,
$$

and consequently

$$
(8) \qquad \|y_n(s)\| \le \eta(s) \quad \text{for a.e. } s \in D \text{ and } n \in N\ ,
$$

where $\eta(s) = K(t,s)(a(s) + b\mu(s))$. From 2^o it is clear that $\eta \in L^1(D,R)$. As $G(u_n)(t) = g(t) + \lambda \int_Q y_n(s)ds \in g(t) + \lambda \text{ mes } Q \cdot \overline{\text{conv}}\, y_n(Q) \subset Y$, from (5) - (8) and Lemma it follows that

$$
\beta_X(V(t)) \le 2\beta_Y(V(t)) \le 2\beta_Y(G(V)(t)) = 2|\lambda|\, \beta_Y(\{\int_Q y_n(s)ds : n \in N\}) \le 2|\lambda| \int_Q \beta_Y(\{y_n(s): n \in N\})ds\ .
$$

Further, in view of (2) we have

$$
\beta_Y(\{y_n(s): n \in N\}) \le 2\beta(\{y_n(s): n \in N\}) \le 2H(t,s)\,\beta(V(s)) \le 2H(t,s)\,\beta_X(V(s))
$$

for $s \in Q$. Hence

$$
(9) \qquad \beta_X(V(t)) \le 4|\lambda| \int_Q H(t,s)\,\beta_X(V(s))ds\ .
$$

By (7) and Lemma the function $s \to \beta_X(V(s))$ is measurable on P and $\beta_X(V(s)) \le \mu(s)$ for a.e. $s \in P$. Thus the function v defined by

$$v(s) = \begin{cases} \beta_X(V(s)) & \text{if } s \in P \\ 0 & \text{if } s \in D \setminus P \end{cases}$$

belongs to $L^2(D,R)$. As (9) holds for every $t \in P$, we have

$$v(t) \le 4|\lambda| \int_D H(t,s)\, v(s)\, ds \quad \text{for a.e. } t \in D.$$

In virtue of the Hölder inequality this implies that

$$v(t) \le 4|\lambda| \|H(t,\cdot)\|_2 \|v\|_2 \quad \text{for a.e. } t \in D,$$

and consequently

$$\|v\|_2 \le 4|\lambda| \|H\|_2 \|v\|_2 .$$

Since $4|\lambda|\|H\|_2 < 1$, this proves that $\|v\|_2 = 0$, i.e. $\beta_X(V(t)) = 0$ for a.e. $t \in P$. On the other hand, in view of 3^o and (4),

$$\lim_{h \to 0} \sup_n \int_D \|u_n(t+h) - u_n(t)\|\, dt = 0 .$$

Therefore, by Lemma,

$$\beta_1(V) \le \int_P \beta_X(V(t))\, dt = 0 ,$$

so that V is relatively compact in $L^1(D,E)$. Moreover, owing to (7), V has equi - absolutely continuous norms in $L^2(D,E)$. From this we deduce that V is relatively compact in $L^2(D,E)$. Applying now Monch's generalization of the Schauder fixed point theorem (cf. [5], we conclude that there exists $u \in B$ such that $u = G(u)$. Clearly u is a solution of (1).

Remark. Let $(t,s,u) \to h(t,s,u)$ be a nonnegative function defined for $0 \le s \le t \le d$, $u \geq 0$, satisfying the following conditions :

(i) for any nonnegative $u \in L^2(D,R)$ there exists the integral $\int_0^t h(t,s,u(s))ds$ for a.e. $t \in D$;

(ii) for any c, $0 < c < d$, $u = 0$ a.e. is the only nonnegative function on $[0,c]$ which belongs to $L^2([0,c], R)$ and satisfies

$$u(t) \le 4 \int_0^t h(t,s,u(s))ds \quad \text{almost everywhere on } [0,c].$$

Combining the proofs of Theorem 2 and Theorem 2 from [6], we can prove the following

Theorem 3. If $1^o - 3^o$ hold and $\beta(f(t,s,Z)) \le$ $\le h(t,s,\beta(Z))$ for $t,s \in D$ and for each bounded subset Z of E, then there exists a subinterval $J = [0,a]$ of D such that the equation

$$x(t) = g(t) + \int_0^t f(t,s,x(s))ds$$

has a solution $x \in L^2(J,E)$.

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RÓWNANIE CAŁKOWE W PRZESTRZENI BANACHA

Streszczenie

W pracy rozpatrujemy równanie całkowe

(1) $$x(t) = g(t) + \lambda \int_D f(t,s,x(s))ds$$

w przestrzeni Banacha E. Pokazujemy, że jeśli operator całkowy generowany przez prawą stronę równania (1) przekształca w sposób ciągły przestrzeń Orlicza $L_\varphi(D,E)$ w siebie oraz spełniony jest warunek Ambrosettiego

$$\alpha(f(t,s,X)) \leq H(t,s)\alpha(X),$$

to dla małych λ istnieje rozwiązanie $x \in L_\varphi(D,E)$ równania (1).