ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY

Problemy Matematyczne 1987 z. 9

STANISŁAW SZYMAŃSKI

WSP w Bydgoszczy

ON THE CATEGORICALNESS OF THE METRICAL GEOMETRY ON THE SPHERE

In our previous paper [4] a system of axioms of the metrical geometry on the sphere is presented and in [5] a theory built on those axiomatics is developed. As the primative notions serve a set S containing at least two distinct elements and the function $d\colon S \times S \to R$ called the distance function. The value $d(A, B)$ will be briefly denoted by AB, the capital letters denote elements of S called points and small letters - real numbers. The distance function verifies the following axioms repeated here for the later use:

A.0. $\bigvee_{A,B} 0 \neq AB \neq \pi$

A.1. $[a \geq 0 \wedge b \geq 0 \wedge (|a - b| \leq AB \leq a+b) \wedge a + b + AB \leq 2\pi] \Leftrightarrow$

$\Leftrightarrow \bigvee_{C} (CA = a \wedge BC = b)$

A.2. $[a > 0 \wedge b > 0 \wedge (|a - b| < AB < a+b) \wedge a + b + AB < 2\pi \Rightarrow$

$\Rightarrow \bigvee_{2C} (AC = a \wedge BC = b)$

A.3. $[A \neq B \wedge AB + BC = AC \wedge \cos AB \cos BD = \cos AD] \Rightarrow$

$\Rightarrow \cos CB \cos BD = \cos CD$.

In the axiom A.2. the sign like $\bigvee_{2C} \ldots$ means that there exist exactly two points C such that ... .

In [4] the following fundamental theorem was formulated:

THEOREM I. A couple $\langle S, d \rangle$ is isomorphic with the basic model $\$_o = \langle S_o, d_o \rangle$ of the metrical geometry on the sphere if and only if the set S and the function d are in compliance with the axioms A.0-A.3. Here S_o is a fixed sphere in the three-dimensional euclidean space, and $d_o(P_1, P_2)$ denotes the spherical length of those points P_1 P_2 on the given sphere S_o.

In [4] only the first half of the above theorem is proved, namely:

THEOREM I A. All axioms A.0-A.1 are satisfied in the basic model $\$_o$

In turn the paper [5] contains the definitions Df1-Df7 of specific notions of the theory developed on the base of our axioms A.0-A.1 and proofs of the basic facts concerning those notions. The purpose of the present paper is to give the proof of the second half of the above-mentioned theorem I, visualy:

THEOREM I B. Each model of axioms A.0 - A.3 is isomorphic with the basic model $\$_o$.

A numeration of theorems in this article will be a continuation of the numeration of theorems T 1-T 42 from the preceding article [5] . However, the notation of the proofs will be more short that in [5]. Each halting-place of the deduction will be noted in the separated line. At the left will be noted the prerequisites and the numbers of quoted theorems, while on the right-hand side we note the conclusions. Those conclusions will be enumerated for the later invocation in others proofs in the sequel. Shortly speaking, the notation of the proofs will be almost formalized. Theorems from [5] and lemmata from [4] will be freely invoked.

T 43. $\bigvee_{E_1, E_2, E_3} [(*)\ 0 < E_1E_2 < \pi \wedge (**)\ E_1 E_2 = E_2 E_3 = \pi/2]$

Proof: From T0, T7 we obtain that $\bigvee_{E_1, E_2} (*)\ 0 < E_1E_2 < \pi$.
Let us denote (1) $a := \pi/2$. Then

(*), (1) (2) $a - a < E_1E_2 < a + a \wedge (3)\ a + a + E_1 E_2 < 2\pi$

(1), (2), (3), A.2 $\Rightarrow \bigvee_{E_3} (**)\ E_1E_2 = E_2 E_3 = \pi/2$.

From (*) and (**) we deduce our thesis.

Before crossing over the proofs of the subsequent theorems we fix some triple of points fulfilling the conditions (*) and (**) and we introduce the designation $P' = \varrho_{E_1 E_2}(P)$. Then we may formulate

T 44 $P'E_3 = \pi/2$

Proof: Directly from the assumption we obtain (1)
$P' = \varrho_{E_1E_2}(P)$

(*), (1), T 38 $\Rightarrow$ (2) $w(E_2, E_1, P')$
(**), Df 3 $\Rightarrow$ (3) $\perp E_2 E_1 E_3$,
(*), (2), (3), T 26 $\Rightarrow$ (4) $\perp P' E_1 E_3$,
(**), (4), Df 3 $P'E_3 = \pi/2$

T 45 $[(1)\ 0 < AB < \pi \wedge (2)\ w(A, B, C_i), (i = 1,2) \Rightarrow \cos C_1C_2 = \cos[\xi_{AB}(C_2) - \xi_{AB}(C_1)]$

Proof: Let us introduce the following designations:

(3) $AB =: c$ (4) $C_1C_2 =: y$, (5) $AC_i =: a_i$, (6) $BC_i = b_i$, $(i = 1,2)$.

(1), T32 $\Rightarrow \bigvee_{x_i} [(7)\ \xi_{AB}(C_i) = x_i, (i=1,2)]$

(1), (2), T 20 $\Longrightarrow$ [(8) $w(A, C_1, C_2) \wedge$ (9) $w(B, C_1, C_2)$]

(8), (4), (5), T 14, L2 $\Longrightarrow$ (10) $1 + 2 \cos a_1 \cos a_2 \cos y - \cos^2 a_1 - \cos^2 a_2 - \cos^2 y = 0$,

(9), (4), (6), T 14, L2 $\Longrightarrow$ (11) $1 + 2 \cos b_1 \cos b_2 \cos y - \cos^2 b_1 - \cos^2 b_2 - \cos^2 a = 0$

(1), (7), (2), T 34 $\Longrightarrow$ (12) $a_i = |x_i| \wedge$ (13) $\cos b_i = \cos (c - x_i)$, $(i = 1,2)$

(12), (13), (10), (11) $\Longrightarrow$ (14) $\sin c \sin (c - x_1 - x_2)[\cos y - \cos (x_1 - x_2)] = 0$

(14), (1), (3) $\Longrightarrow$ (15) $[\cos y = \cos (x_1 - x_2) \vee \sin(c - x_1 - x_2) = 0]$.

If $\sin (c - x_1 - x_2)$ vanish, then we may introduce an additional point C_o such that the following equality holds:

$$C_1C_o = C_2C_o = C_1C_2/ 2$$

and taking into account the pairs (C_1, C_o) and (C_2, C_o) we may reduce the analysis to the case where $\cos y = \cos (x_1 - x_2)$ holds.

T 46 [(1) $\xi_{E_1E_2}(P_i) = \lambda_i \wedge$ (2) $\xi_{P_1' E_3}(P_i) = \varphi_i$, $(i = 1,2)$] $\Longrightarrow$ $\cos P_1P_2 = \cos \varphi_1 \cos \varphi_2 \cos (\lambda_1 - \lambda_1) + \sin \varphi_1 \sin \varphi_2$.

Proof: Assumption : (3) $P_i' = \varsigma_{E_1E_2}(P_i)$,

(3) , T 44 $\Longrightarrow$ $P_i' E_3 = \pi / 2$,

(xx) , (4), T 19 $\Longrightarrow$ [(5) $w(E_i, C_1, C_2) \wedge$ (6) $w(E_1, E_2, P_i')$]

(x), (3), (5), T 41 $\Longrightarrow$ (7) $\perp E_iP_i' P_i$.

(xx) , (4), Df 3 $\Longrightarrow$ (8) $\perp E_iP_i' E_3$

(x) , T2 , T7, T8 $\Rightarrow$ (9) $[0 < E_1 P_1' < \pi \vee 0 < E_2 P_1' < \pi]$,

(9), (7), (8), T 27 $\Rightarrow$ (10) $w(P_1', P_1, E_3)$,

(1), (x), (3), T 39 $\Rightarrow$ (11) $\xi_{E_1 E_2}(P_1') = \lambda_1$,

(9), (10), (7), T 26 $\Rightarrow$ (12) $\perp P_1' P_2' P_2$

(4), T37, T38 $\Rightarrow$ (13) $P_2'' = \varrho_{P_1 E_3}(P_2) \wedge$ (14) $w(P_1', E_3, P_2'')$,

(4), (14), (10), T 20 $\Rightarrow$ (15) $w(P_1', P_1, P_2'')$

(4), (10), (13), T 41 $\Rightarrow$ (16) $P_1 P_2'' P_2 \wedge$ (17) $\perp E_3 P_2'' P_2 \wedge$

$\wedge$ (18) $\perp P_1' P_2'' P_2$,

Let us denote : (19) $\xi_{P_1 E_3}(P_2) = \beta$. Next we have :

(3), (4), (14), (19), T 34 $\Rightarrow$ (20) $P_1' P_2'' = |\beta|$.

(2)(3),(4), (10), T 34 $\Rightarrow$ (21) $P_1' P_1 = \varphi_1$,

(11), (x), (6), T 45 $\Rightarrow$ (22) $\cos P_2' P_1' = \cos(\lambda_2 - \lambda_1)$

$$(16),(17),(18),(20) \Rightarrow (23) \begin{cases} \cos P_1 P_2 = \cos P_1 P_2'' \cos P_2'' P_2 \\ \cos E_3 P_2 = \cos E_3 P_2'' \cos P_2'' P_2 \\ \cos P_1' P_2 = \cos \beta \cos P_2'' P_2 \end{cases}$$

(12), (21), (22), Df 3 $\Rightarrow$ (24) $\cos P_1' P_2 = \cos \varphi_2 \cos(\lambda_2 - \lambda_1)$

(4), (10), (21), T 14 , L 2 $\Rightarrow$ (25) $\cos P_2 E_3 = \pm \sin P_1' P_2 = \sin \varphi_2$,

(4), (14), (20), T 14, L 2 $\Rightarrow$ (26) $\cos P_2'' E_3 = \pm \sin P_1' P_2'' = \sin \beta$,

(2), (4), (10), (14), (19), T 45 $\Rightarrow$ (27) $\cos P_1 P_2'' = \cos(\varphi_1 - \beta)$

From prerequisites (23)-(27) we obtain promised thesis.

T 47 $\bigwedge_{P} \bigvee_{1\lambda} \bigvee_{1\varphi} [\lambda = \xi_{E_1 E_2}(P) \wedge \varphi = \xi_{P' E_3}(P)]$

Proof: (x) , T 32 $\Rightarrow$ (1) $\bigvee_{1\lambda} \lambda = \xi_{E_1 E_2}(P)$,

(x), T 37 $\Rightarrow$ (2) $\bigvee_{P'} P' = \varrho_{E_1 E_2}(P)$

(2), T 44 $\Rightarrow$ (3) $P'E_3 = \pi/2$,

(3), T 32 $\Rightarrow$ (4) $\bigvee_{1\varphi} \varphi = \xi_{P'E_3}(P)$,

From (1) and (4) we deduce our thesis.

T 48 $[(1)\lambda \in (-\pi, \pi\rangle \wedge (2)\varphi \in (-\pi/2, +\pi/2)] \Rightarrow$

$\Rightarrow \bigvee_P = \xi_{E_1E_2}(P) \wedge \varphi = \xi_{P'E_3}(P)]$

Proof:

(x), (1), T 35 $\Rightarrow$ (3) $\bigvee_{P_1} [\lambda = \xi_{E_1E_2}(P_1) \wedge w(E_1, E_2, P_1)]$,

(x x), (3), Df 3 $\Rightarrow$ (4) $\perp E_1E_2E_3 \wedge$ (5) $w(E_1, E_2, P_1)]$

(x) , (4), (5), T26 $\Rightarrow$ (6) $\perp P_1E_2E_3$,

(x x), (6), Df 3 $\Rightarrow$ (7) $P_1E_3 = \pi / 2$,

(2), (7), T 35 $\Rightarrow$ (8) $\bigvee_P [\xi_{P'E_3}(P) = \varphi \wedge w(P_1, E_3, P)]$

(2), (7), (8) $\Rightarrow$ (9) $P_1P < \pi / 2$,

(x x), (7), (8), Df 3 $\Rightarrow$ (10) $\perp E_3P_1E_1 \wedge$ (11) $w(P_1, E_3, P)]$

(7), (10), (11), T26 $\Rightarrow$ (12) $\perp PP_1E_1$

(x) , (4), (9), (12), T 42 $\Rightarrow$ (13) $P_1 = \wp_{E_1E_2}(P)$

(x) , (13), T 37 $\Rightarrow$ (14) $P_1 = P'$

(x) , (13), T 39 $\Rightarrow$ (15) $\xi_{E_1E_2}(P_1) = \xi_{E_1E_2}(P)$

From (3), (8), (14) and (15) we obtain the promised thesis,

T 49 $[(1)\ \lambda = 0 \wedge (2)\ |\varphi| = \pi/2] \Rightarrow \bigvee_P \xi_{E_1E_2}(P) = \lambda \wedge \xi_{P'E_3}(P) = \varphi]$

Proof:

(x) , (1), T 35 $\Rightarrow$ (3) $\bigvee_{P_1} \xi_{E_1E_2}(P_1) = \lambda \wedge w(E_1, E_2, P_1)]$,

(x) , (1), (3), T 34 $\Rightarrow$ (4) $E_1P_1 = 0$

(4) , T8 $\Rightarrow$ (5) $E_1 = P_1$

(x x), (2), (5), T 35 $\Rightarrow$ (6) $\bigvee_P [\xi_{P_1E_2}(P) = w(P_1, P_3, P)]$

(x x), (2), (6), T 34 $\Rightarrow$ (7) $E_1P = \pi / 2$

(x x), Df 3 ==> (8) $\perp E_3E_1E_2$,

(5), (6) ==> (9) w (E_1, E_3, P)

(x x), (8), (9) T 26 ==> (10) $\perp PE_1E_2$,

(10), (7), Df 3 ==> (11) $PE_2 = \pi/2$

(x) , (7), (11), Df 4 , Df 5 ==> (12) $\xi_{E_1E_2}(P) = 0$

(x), T 37 ==> $\bigvee_{P'}$ (13) $P' = \xi_{E_1E_2}(P)$

(x), (1), (12), (13), T 38 ==> (14) $[\xi_{E_1E_2}(P') =$

$= \lambda \wedge w(E_1, E_2, P')$

(x), (1), (3), (14), T 35 ==> (15) $P_1 = P'$

The prerequisites (1), (6), (12) and (15) give already our thesis.

T 50 $[(1)\ \xi_{E_1E_2}(P_1) = \xi_{E_1E_2}(P_2) \cap (2)\ \xi_{P_1' E_3}(P_1) =$

$= \xi_{P_2' E_3}(P_2) \Longrightarrow (P_1 = P_2)$

Proof: Let us denote

(3) $\xi_{E_1E_2}(P_i) = \lambda_i$, (4) $\xi_{P' E_3}(P_i) = \varphi_i$, $(i=1,2)$. We have:

(1), (2), (3), (4) ==> (5) $\lambda_1 = \lambda_2 \wedge$ (6) $\varphi_1 = \varphi_2$

(3), (4), T 46 == (7) $\cos P_1P_2 = \cos\varphi_1 \cos\varphi_2 \cos(\lambda_2 - \lambda_1) +$

$+ \sin\varphi_1 \sin\varphi_2$,

(5), (6), (7) ==> (8) $\cos P_1P_2 = 1$

(8), T7 ==> (9) $P_1P_2 = 0$

(9), T 8 gives already the thesis .

We are now in a position to perform the crucial step in the proof of theorem IB, namely we define the isomorphism between the basic model $\mathfrak{s}_o = \langle S_o, d_o \rangle$ and an arbitrary model $\langle S, d \rangle$ of the system of axioms A.0-A.3. In order to execute such a project let us take under consideration an arbitrary model $\langle S, d\rangle$ and let us fix a triplet of points E_1, E_2, E_3 fulfilling the conditions (x) and (xx) . We need some definition :

Def. I. For each point $P \in S_o$ let $\Phi_o(P)$ denotes the following triplet of real numbers $\langle \cos\lambda \cos\varphi, \sin\lambda \cos\varphi, \sin\lambda\rangle$, where $\lambda = \xi_{E_1E_2}(P)$, $\varphi = \xi_{P' E_3}(P)$.

T II. The above defined Φ_o is a bijection of the set S onto the set S_o.

Proof: From theorems T 47 - T 50 it follows that there exists one-to-one mapping Φ_1 acting between the set S and S_1, where S_1 consists all pairs of reals of the form $\langle \lambda, \varphi\rangle$ $\lambda \in (-\pi, \pi\rangle$ and $\varphi \in (-\pi/2 , \pi/2)$ and the pairs $\langle 0, \pi/2\rangle$, $\langle 0, -\pi/2\rangle$.

In turn the mapping Φ_2 defined by the formula $\Phi_2(\lambda, \varphi) :=$ $:= (x,y,z)$:

$$
(1) \quad \begin{aligned} x &= \cos\lambda \cos\varphi \\ y &= \sin\lambda \cos\varphi \\ z &= \sin\varphi \end{aligned}
$$

maps in a bijective manner the set S_1 onto S_o. Since Φ_1 as well as Φ_2 are bijective, its superposition Φ_0 also must be bijective. The proof of T II is already completed.

In order to complete the proof of the fundamental theorem I B it suffices to demonstrate, that Φ_o transforms the distance function d onto the function d_o defined in the basic model by the following formula:

$$d_o (\langle x_1, y_1, z_1\rangle, \langle x_2, y_2, z_2\rangle) = \arccos (x_1x_2 + y_1y_2 + z_1z_2)$$

So, we shall demonstrate the following theorem:

T III. If (1) $\Phi_o(P_1) = \langle x_1, y_1, z_1\rangle$ and (2) $\Phi_o (P_2) =$

$= \langle x_2, y_2, z_2 \rangle$ then of necessary $P_1P_2 = d_o(\langle x_1, y_1, z_1 \rangle,$ $\langle x_2, y_2, z_2 \rangle)$.

Proof: Let us denote: (3) $\lambda_i := \xi_{E_1E_2}(P_i)$, (4) $\varphi_i :=$ $:= \xi_{P_i'E_3}(P_i)$ for $i \in \{1,2\}$. From (1) - (4) and Df I we obtain

$$(5) \quad \begin{cases} x_i = \cos \lambda_i \cos \varphi_i \\ y_i = \sin \lambda_i \cos \varphi_i \quad (i = 1,2) \\ z_i = \sin \varphi_i \end{cases}$$

(5) , (xxx) $\Longrightarrow$ (6) $d_o(\langle x_1, y_1, z_1 \rangle, \langle x_2, y_2, z_2 \rangle) =$

$= \arccos [\cos \varphi_1 \cos \varphi_2 \cos(\lambda_2 - \lambda_1) + \sin\varphi_1 \sin\varphi_2]$

(3), (4), (6), T 46 give the promised thesis.

From theorems T II and T III it follows that the function Φ_o defined for an arbitrary model of axioms A.0 - A.3 establish a isomorphism of this model with the basic model $\#_o$, so that the theorem I I B is fulfilled and the proof of the categoricalness of the theory built upon the axioms A.0 - A.3 is achieved.

REFERENCES

[1] Borsuk K., Szmielew W., Podstawy geometrii, Warszawa 1970

[2] Dubikajtis L., Geometria metryczna , UMK Toruń, 1971

[3] Mostowski A., Logika matematyczna, Warszawa-Wrocław 1948

[4] Szymański S., A system of axioms of the metrical geometry on the sphere, Problemy Matematyczne 7, WSP Bydgoszcz 1986

[5] Szymański S., On fragments of the metric geometry on the sphere, Problemy Matematyczne 8, WSP Bydgoszcz 1987

KATEGORYCZNOŚĆ GEOMETRII METRYCZNEJ NA SFERZE

Streszczenie

W pracach [4] i [5] podano aksjomatykę oraz rozwinięto teorię geometrii metrycznej na sferze. Podano także dowód, że aksjomatyka ta jest spełniona w pewnym modelu geometrii metrycznej na sferze, nazwanym modelem podstawowym $\mathfrak{S}_0$. Celem niniejszego artykułu jest dalsze rozwinięcie teorii i podanie dowodu jej kategoryczności, tzn. wykazanie, że każdy model naszej teorii jest izomorficzny z modelem podstawowym.