ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY

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WSP w Bydgoszczy

ON A ROHLIN HYPOTHESIS

I. Introduction. Let $\mathcal{C}_n^m$ be the complete m - complex (m-dimensional complex) with n veritices, and $\mathcal{K} \vee L$ be the join of complexes $\mathcal{K}$ and L. The following result of B. Grunbaum [1] is well known:

Theorem A. Let n, k, $n_1, \dots, n_k$ be positive integers, such that $n_1 + \dots + n_k = n + 1$, then the n - complex

$$(1) \quad B^n = \mathcal{C}_{2n_1+1}^{n_1-1} \vee \dots \vee \mathcal{C}_{2n_k+1}^{n_k-1}$$

is not embeddable in $\mathbb{R}^{2n}$.

This result is an extension of results due to Van Kampen [2] and A. Flores [3] on the existence of n - complexes which are not (topologicaly) embeddable in $\mathbb{R}^{2n}$. (For n=1 see [1]).

The present paper is in fact a part of the author's [4] M.S. thesis, written in 1966-67 under the guidance of prof. V.A. Rohlin.

In that time it was not published, because prof. V.A. Rohlin relied on the author, that he would prove the next results :

(B) [Hypothesis of V.A. Rohlin].

Let p_i be the number of the (i-1)-complexes $\mathcal{C}_{2i+1}^{i-1}$ in the decomposition of the complex B^n in the form (1). We shall express this fact writting

$$B^n = B^n(p_1,\dots,p_m) \text{ , where}$$

(2) $$n = \left(\sum_{i=1}^{m} i\, p_i\right) - 1$$

In order that $|B^n(p_1,\dots,p_n)|$ and $|B^n(q_1,\dots,q_k)|$ $(p_m \neq 0,\ q_k \neq 0)$ be homeomorphic it is necessary and sufficient that $m = k,\ p_1 = q_1,\dots,p_m = q_m$.

After 1969, the author was not engaged in this subject. Moreover, the author has learnt about of a more elementary proof of Theorem A than in [4], due to B. Grünbaum [1],[6].

On the other hand the paper of J. Zaks [5] contained the more general results about a minimality property of complexes B^n, than there were in [4]. The only results of [4] (see the Theorem 1 and Corollary 3 below), which are not improved for present time are the results refering to the Rohlin hypothesis.

II. Notations and results

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}$ be the real line, $\mathbb{Z}$ be the subgroup of rational integers. Let $\xi \in \mathbb{R}$. We denote by $[\xi]$ the greatest integer less then or equal to ξ. By $\binom{n}{m}$ we denote as usual $n!/(m!\,(n-m)!)$, where $n \in \mathbb{N}$, $m \in \{0\} \cup \{\mathbb{N}\}$. By π we denote the function: $(\mathbb{N} \setminus \{1\}) \to \mathbb{N}$, $\pi(n) = \text{card}\{x \in \mathbb{N} : 2 \leq x \leq n,\ x \text{ is a prime number}\}$.

If L is a subcomplex of complex K and K is embedded (topologically) in $\mathbb{R}^k$, $|K|$ will denote its realization in $\mathbb{R}^k$, and $|L|$ will be the subset of $|K|$ in the usual way. Let $B^n (n \geq 0)$ be a n-dimensional complex.
By $H_i(B^n)$, $\pi_j(|B^n|)$ we denote, as usual, the i-th homology

group of B^n, the j-th homotopy group of $|B^n|$, $(i=0,1,\ldots,n;$ $j=1,\ldots,n$, $n \geq 0)$.

In the sequel, we shall use the symbol B^n for denoting both

$$B^n=B^n(p_1,\ldots,p_m) \text{ and } B^n=B^n(p_1,\ldots,p_m,\underbrace{0,\ldots,0}_{i \text{ times}}), \quad (i=1,2,\ldots).$$

<u>Proposition 1.</u> Let $B^n=\zeta^n_{2n+3} = B^n(\underbrace{0,\ldots,0,1}_{n \text{ times}})$, $n \geq 1$.

Then

$$H_0(B^n) = \mathbb{Z}, \quad H_1(B^n) = \ldots = H_{n-1}(B^n) = 0 \quad \text{for } n \geq 2,$$

$$H_n(B^n) = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\binom{2n+2}{n+1} \text{ times}} \quad \text{for } n \geq 1.$$

<u>Proof.</u> The case $n = 1$ is trivial.

Let $n \geq 2$. It is not difficult to see that ζ^n_{2n+3} has the homotopy type of the union of $\binom{2n+2}{n+1}$ copies of the n-dimensional sphere. Then if easy to see [7] that

(3) $$H_n(B^n) = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\binom{2n+2}{n+1} \text{ times}}, \quad H_1(B^n) = \ldots = H_{n-1}(B^n) = 0.$$

Finally, by connectedness of $B^n (n > 1)$ we have

(4) $$H_0(B^n) = \mathbb{Z}$$

<u>Corollary 1.</u> Let $B^n=B^n(p_1,\ldots,p_n)$, $p_m \neq 0$, $n \geq 1$. Then

(5) $$H_0(B^n) = \mathbb{Z}, \quad H_1(B^n) = \ldots = H_{n-1}(B^n) = 0 \quad \text{for } n \geq 2;$$

(6) $$H_n(B^n) = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\prod_{i=1}^{m} \binom{2i}{i}^{p_i} \text{ times}} \quad \text{for } n \geq 1.$$

<u>Proof.</u> The case $n=1$ is trivial. Let $B_1^{n_1}$ i $B_2^{n_2}$ are

the complexes of the form (1). Then by using the join operation properties (see, for example, [10], VIII, exercise 5) we obtain:

$$H_{n_1+n_2+1}(B_1^{n_1} \vee B_2^{n_2}) = H_{n_1}(B_1^{n_1}) \otimes H_{n_2}(B_2^{n_2}) .$$

From here and Proposition 1 we get immediately (5) and (6).

Lemma 1. Let $B_1^n = B^n(p_1,\ldots,p_l)$, $B_2^n = B^n(0,\ldots,0,p_r,\ldots,p_k)$, $p_l \neq 0$, $p_r \neq 0$, $(n \geq 1)$. If $r > l$, then

$$H_n(B_1^n) \neq H_n(B_2^n)$$

Proof. From the Proposition 1 one conclude that

$$H_n(B_1^n) = \underbrace{Z \oplus \ldots \oplus Z}_{\prod_{i=1}^{l} \binom{2i}{i}^{p_i} \text{ times}}, \qquad H_n(B_2^n) = \underbrace{Z \oplus \ldots \oplus Z}_{\prod_{j=r}^{k} \binom{2j}{j}^{p_j} \text{ times}},$$

Let (x_n) be a sequence of integers such that $x_0 = \binom{2}{1}$. $x_1 = \binom{4}{2}/\binom{2}{1}, \ldots, x_{n-1} = \binom{2n}{n}/\binom{2n-2}{n-1}, \ldots$. It is easy to see that

(8) $\binom{2i}{i} = \prod_{j=1}^{i} x_{j-1}$, $(i = 1,2,\ldots)$ and

(9) $x_n \uparrow 4$.

By (8) there exists $\xi(i) \in [2, x_i]$ such that

(10) $(\xi(i))^i = \binom{2i}{i}$, $i \geq 1$.

Let $m \in \mathbb{N}$, $t(1),\ldots,t(m) \in (0, +\infty)$, $s(1),\ldots,s(m) \in \mathbb{N}$. It is well know that there exists $h \in [\min_{\mu=1,\ldots,m} t(\mu), \max_{\mu=1,\ldots,m} t(\mu)]$ such that

(11) $\prod_{\mu=1}^{m} (t(\mu))^{s(\mu)} = h^{\sum_{\mu=1}^{m} s(\mu)}$

Therefore this together with (8) and (10) give,

$$\prod_{i=1}^{l} \binom{2i}{i}^{p_i} = \prod_{i=1}^{l} (\xi(i))^{ip_i} = (h_1)^{\sum_{i=1}^{l} ip_i}$$

$$\prod_{j=1}^{k} \binom{2j}{j}^{p_j} = \prod_{j=r}^{k} (\xi(j))^{jp_j} = (h_2)^{\sum_{j=r}^{k} jp_j}$$

From (9) and (10) one conclude that if $r > 1$ we have $\xi(r) > \xi(l)$.

Since $\sum_{i=1}^{l} ip_i = \sum_{j=r}^{k} jp_j = n + 1$ and

$h_1 \in [\xi(1), \xi(l)]$, $h_2 \in [\xi(r), \xi(k)]$ we infer that

$$\prod_{j=r}^{k} \binom{2j}{j}^{p_j} = \prod_{i=1}^{l} \binom{2i}{i}^{p_i} .$$

This together with the Proposition 1 give the assertion.

Definition 1. Let $B_1^n = B^n(p_1,\dots,p_m)$ and $B_2^n = B^n(q_1,\dots,q_m)$ be n-complexes of the form (1), $n \geq 1$. Let $f_i = \min\{p_i, q_i\}$, $\tilde{p}_i = p_i - f_i$, $\tilde{q}_i = q_i - f_i$, $i = 1,\dots,m$.

If not all $f_1,\dots,f_m$ are zero then the k-complexes $\tilde{B}_1^k = B^k(\tilde{p}_1,\dots,\tilde{p}_m)$, $\tilde{B}_2^k = B^k(\tilde{q}_1,\dots,\tilde{q}_m)$ are called reciprocal simple parts of B_1^n, B_2^n. Thus we have

(12) $B_1^n = B^t \vee \tilde{B}_1^k$, $B_2^n = B^t \vee \tilde{B}_2^k$, where $B^t = B^t(f_1,\dots,f_m)$.

If all $f_1,\dots f_m$ are zero then the complexes B_1^n, B_2^n are called reciprocal simple. (For example, the complexes B_1^n, B_2^n from Lemma 1 are the reciprocal simple complexes).

Lemma 2. Let $B_1^n = B^n(p_1,\dots,p_m)$, $B_2^n = B^n(q_1,\dots,q_r)$ be the reciprocal simple complexes, $p_m \neq 0$, $q_r \neq 0$.

Let

(i) $r < (3/4)\,m$, if m is even,

(ii) $r < [(3/4)m]$, if m is odd.

If $m \geq 12$, then

$$H_n(B_1^n) \neq H_n(B_2^n)$$

Proof. By the factorization of the number $\prod_{j=1}^{r} \binom{2j}{j}^{q_j}$ into prime factors, using Bertrand's postulate (proved by P.L. Chebyshev), saying that for any $k \geq 2$ there exists such a prime number p that $k < p < 2k$ (see, for example, [8] p.134), it is easy to see that the greatest prime number is in the factorization of $\binom{2r}{r}$.

Next we want to establish that by (i) or (ii) there exists a prime number in the factorization of the number $\binom{2m}{m}$ which belongs to $(2r, 2m)$. Since this fact we would have in the factorization of $\binom{2m}{m}$ into prime factors a prime number which is absent in the factorization of $\binom{2r}{r}$. Using Proposition 1 this would end the proof.

Let $n \in \mathbb{N} \setminus \{1\}$. If is well know the result of Chebyshev:

$$(13) \quad 0{,}92 \frac{n}{\ln n} < \pi(n) < 1.11 \frac{n}{\ln n} .$$

Assume that m is even, i.e. exists $k \in \mathbb{N}$ such that $m = 2k$. Hence by (i) we get

$$\pi(2m) - \pi(2r) > \pi(4k) - \pi(3k) > 0.92 \frac{4k}{\ln 4k} - 1.11 \frac{3k}{\ln 3k} = \frac{0{,}35\, k\, (\ln k - 1.67)}{(\ln 4k) \cdot (\ln 3k)} .$$

If $k \geq 6$, then $\ln k > 1.67$, and therefore $\pi(2m) - \pi(2r) > 0$.

Now assume that m is odd i.e. exists $k \in \mathbb{N}$ such that $m = 2k + 1$. Note that $\left[\frac{3}{2}(2k + 1)\right] - 1 = 3k$. Hence if

$k \geq 6$, then

$$\pi(2m) - \pi(2r) = \pi(4k+2) - \pi(2r) = \pi(4k+2) - \pi(3k)$$

$$> 0.92 \frac{4k+2}{\ln(4k+2)} - 1,11 \frac{3k}{\ln 3k} > 0.92 \frac{4k}{\ln 4k} -$$

$$- 1,11 \frac{3k}{\ln 3k} > 0.$$

Remark 1. Let $B_1^n = B^n(p_1,\dots,p_s)$, $B_2^n = B^n(q_1,\dots,q_k)$, $n \geq 1$, $p_s \neq 0$.

Let $s = 2, 3, 4, 5, 6, 7, 9, 10, 12, 13$. If $s > k$, then $H_n(B_1^n) \neq H_n(B_2^n)$. This follows from Lemma 1, applying the factorization of $\binom{2s}{s}$. In the same way we can obtain the next result:

If $n \leq 21$, then there exist exactly three pairs (for $s=8, 11, 14$) of the reciprocal simple complexes of the form (1), which have the same homology groups. These are

$$(14) \quad \begin{cases} B_1^{16} = B^{16}(\underbrace{4, 0,0,0,1,0,0,1}_{s=8}) \\ B_2^{16} = B^{16}(\underbrace{0, 3,0,1,0,0,1}_{k=7}) \end{cases}$$

$$(15) \quad \begin{cases} B_1^{20} = B^{20}(\underbrace{0,2,0,0,0,1,0,0,0,0,1}_{s=11}) \\ B_2^{20} = B^{20}(\underbrace{1,0,0,0,2,0,0,0,0,1}_{k=10}) \end{cases}$$

$$(16) \quad \begin{cases} B_1^{21} = B_1^{21}(\underbrace{1,0,0,1,0,0,\dots,0,1}_{s=14}) \\ B_2^{21} = B_2^{21}(\underbrace{0, 3,1,0,\dots,0,1}_{k=13}) \end{cases}$$

Theorem 1. Let $B_1^n = B^n(p_1,\dots,p_k)$, $B_2^n = B^n(q_1,\dots,q_k)$ be the n-complexes of the form (1). Then $|B_1^n|$ and $|B_2^n|$

are nonhomeomorphic, provided

(i) $p_u \neq 0, \quad q_u = 0 \quad$ or $\quad p_u = 0, \quad q_u \neq 0$

for some $u \in \{1,2,\dots,k\}$,

or

(ii) the reciprocal simple parts $\tilde{B}_1^k$, $\tilde{B}_2^k$ of B_1^n, B_2^n are such that

$$H_k(\tilde{B}_1^k) \neq H_k(\tilde{B}_2^k)$$

Proof. (i) We shall begin with the case where $p_u \neq 0$, $q_u = 0$. (The proof in the case $p_u = 0$, $q_u \neq 0$ is analogous). Then there exists a factor $\mathcal{C}_{2u+1}^{u-1}$ in the decomposition of B_1^n. Denote by $\mathcal{L}_u$ the set of vertice of $\mathcal{C}_{2u+1}^{u-1}$. Let σ^{n-1} be a (n-1)-dimensional simplex of B_1^n such that σ^{n-1} contains exactly $(u-1)$ vertices of $\mathcal{L}_u$. By the simplicial structure of B_1^n the simplex σ^{n-1} is a face of exactly u n-dimensional simplexes of B_1^n. Let a point $a_0 \in \operatorname{Int} |\sigma^{n-1}|$. Then the n-th local group is $\underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{(u-1)\text{times}}$.

Let $P_i \subset |B_i^n|$, $(i = 1,2)$ be a set of points for which the n-th local group is $\underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{(u-1)\text{times}}$. Clearly, we get $\dim P_1 = n-1$, $\dim P_2 \leq n-2$, because in B_2^n there is no (n-1)-simplex, which is a face of exactly $(u-1)$ n-simplexes. Finally, recall that the n-th local group is a topological invariant. This completes the proof.

The proof in the case (ii) follows immediately from Corollary 1.

From the Theorem we get immediately two results:

Corollary 2. Let B_1^n, B_2^n be the reciprocal simple complexes. Then $|B_1^n|$ is not homeomorphic to $|B_2^n|$.

Corollary 3. Let B_1^n, B_2^n be the n - complexes of the form (1). Let for B_1^n and B_2^n the conditions of Lemma 1 or Lemma 2 be satisfied. Then $|B_1^n|$ is not homeomorphic to $|B_2^n|$.

Remark 3. Let $B_1^n = B^n(p_1,\dots,p_t)$, $B_2^n = B^n(q_1,\dots,q_t)$ be the n-complexes. Let $n \leq 37$. If $\sum_{i=1}^{t} |p_i - q_i| \neq 0$ then $|B_1^n|$ is not homeomorphic to $|B_2^n|$.

Indeed, we can assume that the condition of Theorem 1 does not hold. Therefore we can restrict to the case $p_t \neq 0$, $q_t \neq 0$ and $(p_i q_i = 0) \Longleftrightarrow (p_i = q_i = 0)$ for all $i < t$.
Let $\tilde{B}^k = B^k(\tilde{p}_1,\dots,\tilde{p}_s)$, $\tilde{B}_2^k = B^k(\tilde{q}_1,\dots,\tilde{q}_r)$ be the reciprocal simple parts of B_1^n, B_2^n $(\tilde{p}_s \neq 0,\ \tilde{q}_r \neq 0)$. Since $s \neq r$, then without loss of generality we can assume $s > r$. Let B^m be a m-complex of the form (1) for which $B_1^n = B^m \vee \tilde{B}_1^k$, $B_2^n = B^m \vee \tilde{B}_2^k$. By our assumption it is easy to see that if $\max\{\tilde{p}_i,\tilde{q}_i\} > 0$ then $\min\{p_i,q_i\} \neq 0$ $(i=1,\dots,s)$ and

$$(17)\qquad m \geqslant \sum_{i=1}^{s} i \cdot \operatorname{sign}(\max\{p_i, q_i\}) \geqslant r+s .$$

Let $s \leq 14$. By Remark 1, only if $s \in \{8,11,14\}$ the condition (ii) of Theorem 1 does not hold. If $s = 8$, then by Remark 1 $H_k(\tilde{B}_1^k) = H_k(\tilde{B}_2^k)$ only for $k \geqslant 16$. If $k \leq 21$, then from view of B_1^{16}, B_2^{16} (see Remark 1) and from (17) we get : $m \geqslant 28$. Hence $n \geqslant k + m+1 = 16 + 28 + 1 = 45$. If $k \geqslant 22$, then by (17) $m \geqslant s + r = 8 + 7$, Because the equality $H_k(\tilde{B}_1^k) = H_k(\tilde{B}_2^k)$ holds only for $r = 7$.
Therefore $n \geqslant 22 + 15 + 1 = 38$.

If $s = 11$ or $s = 14$, we can consider in the same way that $n \geqslant 45$.

If $s = 15$, then the factorization of $\binom{30}{15}$ contains the prime number 29. By $r < s$ and by Proposition 1 we get $H_k(\tilde{B}_1^k) \neq H_k(\check{B}_2^k)$, which contradicts our assumption.

If $s \geqslant 16$, then by Corollary 3 we have $r \geqslant 12$. By (17) $m \geqslant s + r \geqslant 16 + 12 = 28$. Hence $n \geqslant m + k + 1 \geqslant m + s + 1 \geqslant 28 + 16 + 1 = 45$, so the proof is finished.

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O PEWNEJ HIPOTEZIE ROCHLINA

Streszczenie

Niech $\mathscr{C}^{n-1}_{2n+1}$ będzie (n-1)- wymiarowym szkieletem 2n-wymiarowego simpleksu.

Niech $|K \vee L|$ będzie złączeniem (join) wielościanów $|K|$ i $|L|$. Oznaczmy następnie

$$(*) \quad |B^m(p_1,\dots,p_k)| = \underbrace{\mathscr{C}^0_1 \vee \dots \vee \mathscr{C}^0_1}_{p_1} \vee \underbrace{\mathscr{C}^1_3 \vee \dots \vee \mathscr{C}^1_3}_{p_2} \vee \dots \underbrace{\vee \mathscr{C}^{k-1}_{2k+1} \vee \dots \vee \mathscr{C}^{k-1}_{2k+1}}_{p_k},$$

gdzie m jest wymiarem wielościanu $|B^m(p_1,\dots,p_k)|$, a

$$1 + m = \sum_{i=1}^{k} ip_t, \quad (p_k \neq 0).$$

Wiadomo (Tw. B. Grunbauma 1967), że wielościan type (*) wymiaru m nie daje się topologicznie zanurzyć w $\mathbb{R}^{2m}$.

Z drugiej strony wielościany typu (*) posiadają szereg interesujących własności dotyczących np. minimalności. Badania w szczególnych przypadkach były zainicjowane pracami K. Kuratowskiego (1931), A. Floresa (1932/33) i Van Kampena (1932). W 1967 r. W.A. Rochlin sformułował hipotezę, że dwa wielościany $|B^m(p_1,\dots,p_n)|$ i $|B^m(q_1,\dots,q_s)|$ są homeomorficzne wtedy i tylko wtedy, gdy $k = s$, $p_1 = q_1,\dots,p_k=q_k$. W niniejszym artykule podano szereg wyników dotyczących tej hipotezy.

В.П.Одинец, ОБ ОДНОЙ ГИПОТЕЗЕ РОХЛИНА

Резюме

Пусть $\mathcal{B}^{n-1}_{2n+1}$ - $(n-1)$-мерный остов $2n$-мерного симплекса. Пусть $|K \vee L|$ джойн полиэдров $|K|$ и $|L|$. Пусть

$$(*)\quad |B^m(p_1,\dots,p_k)| = |\underbrace{\mathcal{B}_1^0 \vee\dots\vee \mathcal{B}_1^0}_{p_1} \vee \underbrace{\mathcal{B}_3^1 \vee\dots\vee \mathcal{B}_3^1}_{p_2} \vee\dots\vee \underbrace{\mathcal{B}_{2k+1}^{k-1} \vee\dots\vee \mathcal{B}_{2k+1}^{k-1}}_{p_k}|$$

где m размерность полиэдра $|B^m(p_1,p_2,\dots,p_k)|$, $(m = (\sum_{i=1}^{k} i p_i) - 1)$, $p_k \neq 0$. Известно, (теорема Бранко Грюнбаума (1967)), что полиэдр $(*)$ размерности m не вкладывается топологически в $\mathbb{R}^{2m}$. С другой стороны полиэдры типа $(*)$ обладают рядом замечательных свойств, например, минимальности. Их изучение в частных случаях восходит к работам К.Куратовского (1931), А.Флореса (1932/1933), Ван Кампена (1932).

В 1967 г. В.А.Рохлин высказал гипотезу, что два полиэдра $|B^m(p_1,p_2,\dots,p_k)|$ и $|B^m(q_1,q_2,\dots,q_s)|$ гомеоморфны тогда и только тогда, когда $k=s$, $p_1=q_1,\dots,p_k=q_k$.

В настоящей работе приведен ряд результатов в обоснование этой гипотезы.