ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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ON CONTINUITY AND SEMICONTINUITY POINTS. FUNDAMENTAL THEOREMS

R denotes the real line and Q the set of rationals. Let $(X, \mathcal{T})$ be a topological space and $f: X \to R$ be a real function. The purpose of the present paper is to study the set of all points at which f is continuous, the set of all points at which f is semicontinuous and the relation between these sets.

We say that f is upper semicontinuus at a point $x \in X$ iff

$$\forall \varepsilon > 0 \quad \exists U \in \mathcal{T} \quad (x \in U \ \& \ f * U \subseteq (-\infty, f(x) + \varepsilon))$$

and f is lower semicontinuous at x iff

$$\forall \varepsilon > 0 \quad \exists U \in \mathcal{T} \quad (x \in U \ \& \ f * U \subseteq (f(x) - \varepsilon, \infty)).$$

Recall that f is continuous at x iff f is lower and upper semicontinuous at x. ([E] p.86).

We use the following notation. If the set $A = \{y \in R: \exists U \in \mathcal{T} \ (x \in U \ \& \ f * (U - \{x\}) \subseteq (-\infty, y))\}$ is non-empty then $\mathcal{T}\text{-}\lim\sup_{t \to x} f(t) = \inf A$. If $A = \emptyset$ then $\mathcal{T}\text{-}\lim\sup_{t \to x} f(t) = \infty$.

If the set $B = \{y \in R: \exists U \in \mathcal{T} \ (x \in U \ \& \ f * (U - \{x\}) \subseteq (y, \infty))\}$ is non-empty then $\mathcal{T}\text{-}\lim\inf_{t \to x} f(t) = \sup B$. If $B = \emptyset$ then $\mathcal{T}\text{-}\lim\inf_{t \to x} f(t) = -\infty$. $C(f)$ is the set of all points at which f is continuous. It is easy to show that

$C(f) = \{x \in X: \mathcal{T}\text{-}\lim\inf_{t \to x} f(t) = \mathcal{T}\text{-}\lim\sup_{t \to x} f(t) = f(x)\}$.

$S(f)$ is the set of all points at which f is upper semicontinuous. Thus $S(f) = \{x \in X: \mathcal{T}\text{-}\lim\sup_{t \to x} f(t) \leq f(x)\}$.

$S'(f)$ is the set of all points at which f is lower semicontinuous.

$$S'(f) = \{x \in X: \mathcal{T}\text{-}\lim\inf_{t \to x} f(t) \geq f(t)\}.$$

Let $B(f) = \{x \in X: \mathcal{T}\text{-}\lim\inf_{t \to x} f(t) = \mathcal{T}\text{-}\lim\sup_{t \to x} f(t)\}$,

$T(f) = \{x \in X: \mathcal{T}\text{-}\lim\sup_{t \to x} f(t) < f(x)\}$,

$T'(f) = \{x \in X: \mathcal{T}\text{-}\lim\inf_{t \to x} f(t) > f(x)\}$.

Notice that

a) $C(f) = S(f) \cap S'(f)$,

b) $C(f) \cap T(f) = C(f) \cap T'(f) = \emptyset$,

c) $T(f) \subseteq S(f)$ and $T'(f) \subseteq S'(f)$,

d) $C(f) \subseteq B(f) \subseteq C(f) \cup T(f) \cup T'(f)$,

e) since $\mathcal{T}\text{-}\lim\sup_{t \to x} f(t) = -(\mathcal{T}\text{-}\lim\inf_{t \to x} (-f)(t))$,

$S(f) = S'(-f)$ and $S'(f) = S(-f)$.

LEMMA 0. If functions $f, g\ X \to R$ are bounded then

a) $\mathcal{T}\text{-}\lim\sup_{t \to x} (f+g)(t) \leq \mathcal{T}\text{-}\lim\sup_{t \to x} f(t) + \mathcal{T}\text{-}\lim\sup_{t \to x} g(t)$,

b) $\mathcal{T}\text{-}\lim\inf_{t \to x} (f+g)(t) \geq \mathcal{T}\text{-}\lim\inf_{t \to x} f(t) + \mathcal{T}\text{-}\lim\inf_{t \to x} g(t)$,

c) $\mathcal{T}\text{-}\lim\sup_{t \to x} (f+g)(t) \geq \mathcal{T}\text{-}\lim\sup_{t \to x} f(t) + \mathcal{T}\text{-}\lim\inf_{t \to x} g(t)$,

d) $\mathcal{T}\text{-}\lim\inf_{t \to x} (f+g)(t) \leq \mathcal{T}\text{-}\lim\inf_{t \to x} f(t) + \mathcal{T}\text{-}\lim\sup_{t \to x} g(t)$, LEMMA 1. If a series $\sum_{n\in N} f_n(t)$ is uniformly convergent then

a) $\mathcal{T}\text{-}\lim_{t\to x} \sup (\sum_{n\in N} f_n)(t) \le \sum_{n\in N} \mathcal{T}\text{-}\lim_{t\to x} \sup f_n(t)$,

b) $\mathcal{T}\text{-}\lim_{t\to x} \inf (\sum_{n\in N} f_n)(t) \ge \sum_{n\in N} \mathcal{T}\text{-}\lim_{t\to x} \inf f_n(t)$,

c) $\bigcap_{n\in N} C(f_n) \subseteq C(\sum_{n\in N} f_n)$.

These lemmas are well-known in the case if X is a metric space [L].

The proofs for topological and for metric spaces are similar.

PROPOSITION 0. $C(f)$ is a G_δ set for every function $f: X \to R$. (for metric spaces X see [SI] p. 121).

Proof. If $x \in C(f)$ then for every $n \in N$ there exists a neighbourhood $U(x,n)$ of x such that $f * U(x,n) \subseteq (f(x) - \frac{1}{n}, f(x) + \frac{1}{n})$. Let $V(n) = \bigcup \{U(x,n): x \in C(f)\}$. Then $V(n) \in \mathcal{T}$ for each $n \in N$ and $C(f) = \bigcap_{n\in N} V(n)$.

Indeed, assume that $x \in \bigcap_{n\in N} V(n)$. Then

$$\forall n\in N \quad \exists y \in C(f) \quad x \in U(y,2n).$$

Since $|f(x)-f(z)| < \frac{1}{n}$ for each $z \in U(y,2n)$, we have

$$\forall n\in N \quad \exists U\in \mathcal{T} \quad (x \in U \;\&\; f * U \subseteq (f(x) - \tfrac{1}{n}, f(x) + \tfrac{1}{n})).$$

Thus $x \in C(f)$. The inclusion $C(f) \subseteq \bigcap_{n\in N} V(n)$ is clear.

PROPOSITION 1. Assume that there exist dense sets $K, L \subseteq X$ such that $K \cup L = X$ and $K \cap L = \emptyset$. Then for every G_δ set D there exists a function $f: X \to R$ such that $C(f) = D$. (for $X = R$ see [O] Th. 7.1, 7.2, [S] p. 3)

Proof. Let $(a_n)_{n\in N}$ be a sequence of positive reals such that $\sum_{n\in N} a_n = 1$ and $a_n = 2\sum_{k\ge n+1} a_k$ $(a_n = 2\cdot 3^{-n})$.

Let $(G_n)_{n\in N}$ be a sequence of open sets such that $G_{n+1} \subseteq G_n$ for $n = 1,2,\ldots$ and $D = \bigcap_{n\in N} G_n$.

For every $n \in N$ we define the function $f_n : X \to R$

$$f_n(x) = \begin{cases} 0 & \text{for } x \in G_n, \\ a_n & \text{for } x \in K - G_n, \\ -a_n & \text{for } x \in L - G_n. \end{cases}$$

Then $G_n \subseteq C(f_n)$. Notice that $C(f_n) = G_n$. Indeed, if $x \notin G_n$ then $x \in K$ and $f_n(x) > \mathcal{T}\text{-}\lim\inf_{t\to x} f_n(t)$ or $x \in L$ and $f_n(x) < \mathcal{T}\text{-}\lim\sup_{t\to x} f_n(t)$.

Let $f(x) = \sum_{n\in N} f_n(x)$.

Since $D \subseteq \bigcap_{n\in N} C(f_n)$ and f is the sum of a uniformly convergent series ($|f_n(x)| < a_n$ for each $x \in X$), we have $D \subseteq C(f)$.

Assume that $x \in K-D$ and $n' = \min\{n : x \notin G_n\}$. Then

$\mathcal{T}\text{-}\lim\inf_{t\to x} f(t) = \sum_{n<n'} f_n(x) + \mathcal{T}\text{-}\lim\inf_{t\to x} \sum_{n\geqslant n'} f_n(t) \leq$

$\mathcal{T}\text{-}\lim\inf_{t\to x} f_{n'}(t) + \sum_{n>n'} \mathcal{T}\text{-}\lim\sup_{t\to x} f_n(t) \leq \frac{1}{2} a_{n'}$, and

$f(x) \geqslant a_{n'}$. Hence $x \notin C(f)$. Similarly, $(L-D) \cap C(f) = \emptyset$.

REMARK. The assumption that there exist dense, disjoint subsets $K, L \subseteq X$ is essential. Example: if $X = \{0\}$ then $C(f) \neq \emptyset$ for every function $f : X \to R$.

LEMMA 2. Assume that $(X, \mathcal{T})$ is a T_1 space. Then for every $f : X \to R$ and for each $a \in R$ the sets

$$A = \{x \in X : \mathcal{T}\text{-}\lim\sup_{t\to x} f(t) < a\}$$

and

$$A' = \{x \in X : \mathcal{T}\text{-}\lim\inf_{t\to x} f(t) > a\}$$ are open.

Proof. If $x \in A$ then there is a neighbourhood U of x such that $f(U-\{x\}) \subseteq (-\infty, a)$. Since $\{x\}$ is closed, $U-\{x\}$ is a neighbourhood of every point $y \in U-\{x\}$. Hence $x \in U \subseteq A$ and

A is open.

Since $A' = \{x \in X: \mathcal{T}\text{-}\lim\limits_{t \to x} \sup (-f)(t) < -a\}$, we have $A' \in \mathcal{T}$.

PROPOSITION 2. If $(X,\mathcal{T})$ is T_1 space then $B(f)$ is a G_δ set for every $f: X \to R$.

Proof. If $x \notin B(f)$ then there are $p,q \in Q$ such that

$$\mathcal{T}\text{-}\lim_{t \to x} \inf f(t) \leq p < q \leq \mathcal{T}\text{-}\lim_{t \to x} \sup f(t).$$

Let $B(p,q) = \{x \in X: \mathcal{T}\text{-}\lim\limits_{t \to x} \inf f(t) \leq p < q \leq \mathcal{T}\text{-}\lim\limits_{t \to x} \sup f(t)\}$.

Then $X\text{-}B(f) = \bigcup\limits_{p,q \in Q} B(p,q)$ and

$$B(p,q) = X - (\{x \in X: \mathcal{T}\text{-}\lim_{t \to x} \sup f(t) < q\} \cup \{x \in X: \mathcal{T}\text{-}\lim_{t \to x} \inf f(t) > p\}).$$

Thus the sets $B(p,q)$ are closed and $X\text{-}B(f)$ is a F_σ set.

REMARK. The coverse theorem is false. Example: let $(X,\mathcal{T})$ be R with the qualitative topology. Let D be a $\mathcal{T}\text{-}G_\delta$ set such that D is not G_δ in the Euclidean topology. Then $B(f) \neq D$ for every $f: X \to R$. ([N3], [N4] pp. 13–14)

Recall that $(X,\mathcal{T})$ is a Baire space iff every open, non-empty set $U \subseteq X$ is of second category in X.

PROPOSITION 3. Let $(X,\mathcal{T})$ be a Baire space and $\emptyset \neq U \in \mathcal{T}$. If $U \subseteq S(f)$ or $U \subseteq S'(f)$ then $U \cap C(f) \neq \emptyset$.

Proof. We use the following theorem of Fort [F]:

Theorem. For every function upper (lower) semicontinuous $f: X \to R$ there exist open and dense sets $G_n \subseteq X (n \in N)$ such that $\bigcap\limits_{n \in N} G_n \subseteq C(f)$.

Let $U \in \mathcal{T}$ and $f: X \to R$ be a function. We use the theorem of Fort for sub-space U of X and for the function $f|U$. Then there is $x \in U \cap C(f|U)$. Since $U \in \mathcal{T}$, we have $x \in C(f)$.

COROLLARY. If $(X, \mathcal{T})$ is a Baire space then $\operatorname{int}(S(f) - C(f)) = \emptyset$.

REMARK. The assumption "X is a Baire space" is essential.

Example: let $X = R$ and $\mathcal{T} = \{G - I$: G is open in the Euclidean topology, I has Lebesgue measure zero$\}$. There exists $f: X \to R$ such that $S(f) = R$ and $C(f) = \emptyset$ [N2].

PROPOSITION 4. If $\operatorname{int}(A) = \emptyset$ then there exists a function $f: X \to R$ such that $S(f) - C(f) = A$ $(S'(f) - C(f) = A)$.

Proof. It is straigh-forward to see that the function

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \notin A \end{cases} \quad (\chi_{X-A}) \text{ has the desired properties.}$$

PROPOSITION 5. Assume that there exist dense, pairwise disjoint sets K_n $(n \in N)$ and $X = \bigcup_{n \in N} K_n$. Then for every set $A \subset X$ there is a function $f: X \to R$ such that $S(f) = A$ $(S'(f) = A)$.

Proof.

Let

$$f(x) = \begin{cases} 1 & \text{for } x \in A - \operatorname{int} A, \\ 0 & \text{for } x \in \operatorname{int} A, \\ \frac{1}{n} & \text{for } x \in K_n - A,\ n = 1, 2, \ldots \end{cases}$$

Then $S(f) = A$. Indeed, if $x \in \operatorname{int}(A)$ then $x \in C(f)$. If $x \in A - \operatorname{int}(A)$ then $x \in S(f)$.

Let $x \in K_n - A$. Since the sets K_n are dense, we have

$$\mathcal{T}\text{-}\limsup_{t \to x} f(t) \geq 0 > f(x) \quad \text{and} \quad x \notin S(f).$$

PROPOSITION 6. For every function $f: X \to R$ the set $T(f)$ $(T'(f))$ is a sum of countably many isolated sets.

Proof. If $x \in T(f)$ then there exists $q(x) \in Q$ such that

$$f(x) > q(x) > \mathcal{T}\text{-}\limsup_{t \to x} f(t).$$

Let $T(f,q) = \{x \in T(f) : q = q(x)\}$ for each $q \in Q$.

The set $T(f,q)$ is isolated. Indeed, if $x \in T(f,q)$ then $\mathcal{J}\text{-}\limsup_{t \to x} f(t) < q$ and there is a neighbourhood U of x such that $f(U - \{x\}) \subseteq (-\infty, q)$. Thus $U \cap T(f,q) = \{x\}$.

Notice that $T(f) = \bigcup_{q \in Q} T(f,q)$.

LEMMA 3. Let $C = \bigcup_{n \in N} C(n)$, $C(n)$ are isolated and $C(n) \cap C(m) = \emptyset$ for $n \neq m$. There are sets $C(i,j)$ ($i,j \in N$ and $j \leq 2^{i-1}$) which have the following properties:

(0) $C = \bigcup\{C(i,j) : i,j \in N,\ j \leq 2^{i-1}\}$,

(1) $C(i,j) \cap C(k,l) = \emptyset$ for $(i,j) \neq (k,l)$,

(2) $C(n) = \bigcup\{C(n,j) : j = 1,2,\ldots,2^{n-1}\}$,

(3) $C(n,m) \cap \mathrm{cl}(\bigcup\{C(l,k) : (2k-1)\cdot 2^{-l} > (2m-1)\cdot 2^{-n}\}) = \emptyset$.

Proof.

Let $g: \{(i,j) : i,j \in N,\ j \leq 2^{i-1}\} \xrightarrow{1-1} Q$

$$g(i,j) = (2j-1)\cdot 2^{-i}.$$

We define inductively partitions of the sets $C(n)$. Let $C(1,1) = C(1)$. Assume that there is a partition $C(k,j)$, $j = 1,2,\ldots,2^{k-1}$ (it is possible that $C(k,j) = \emptyset$ for some j) of the set $C(k)$ for every $k < n$ such that

(i) if $g(k,j) < g(t,l)$ then $C(k,j) \cap \mathrm{cl}\,C(t,l) = \emptyset$

and

(ii) if $g(k,j) < g(t,l)$ and $t > k$ then there is $s \geq j$ such that $C(t,l) \subseteq \mathrm{cl}\, C(k,s)$.

The sets $C(n,1), C(n,2),\ldots,C(n,2^{n-1})$ are defined as follows

$$C(n,1) = C(n) - \bigcup_{i=1}^{2^{n-1}-1} \mathrm{cl}\, C(g^{-1}(i \cdot 2^{-n+1})),$$

$$C(n,2) = C(n) \cap \mathrm{cl}\, C(g^{-1}(2^{-n+1})) - \bigcup_{i>1}^{2^{n-1}-1} \mathrm{cl}\, C(g^{-1}(i \cdot 2^{-n+1})),$$

$\vdots$

$$C(n,k) = C(n) \cap \mathrm{cl}\ C(g^{-1}((k-1)\ 2^{-n+1}) - \bigcup_{i>k}^{2^{n-1}-1} \mathrm{cl}\ C(g^{-1}(i \cdot 2^{-n+1})),$$

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$$C(n,2^{n-1}) = C(n) \cap \mathrm{cl}\ C(g^{-1}((2^{n-1}-1) \cdot 2^{-n+1})).$$

Notice that $C(n,j) \subseteq C(n) - \bigcup \{\mathrm{cl}\ C(k,i) : k \leq n \ \& \ g(k,i) > g(n,j)\}$ for $j \leq 2^{n-1}$.

Let $g(k,j) < g(n,l)$. We shall prove that $C(n,l) \subseteq \mathrm{cl}\ C(k,t)$ for some $t \geq j$. We have $C(n,l) \subseteq \mathrm{cl}\ C(g^{-1}((l-1) \cdot 2^{-n+1}))$ and $(l-1) \cdot 2^{-n+1} = (2l-2) \cdot 2^{-n} = \max \{g(k,j) : k \leq n \ \& \ g(k,j) < g(n,l)\}$.

Thus $g(k,j) \leq (l-1) \cdot 2^{-n+1}$. If $g(k,j) < (l-1) \cdot 2^{-n+1}$ then by inductionnal assumption we have $C(g^{-1}((l-1) \cdot 2^{-n+1})) \subseteq \mathrm{cl}\ C(k,t)$.

Hence $C(n,l) \subseteq \mathrm{cl}\ C(g^{-1}((l-1) \cdot 2^{-n+1})) \subseteq \mathrm{cl}\ C(k,t)$.

If $g(k,j) = (l-1) \cdot 2^{-n+1}$ then $C(n,l) \subseteq \mathrm{cl}\ C(k,j)$.

Now we shall prove that $C(k,j) \cap \mathrm{cl}\ C(t,l) = \emptyset$ if $g(k,j) < g(t,l)$. Let $k < n$ and $g(k,j) < g(n,i)$. Then $C(n,i) \subseteq \mathrm{cl}\ C(g^{-1}((i-1) \cdot 2^{-n+1}))$ and $g(k,j) \leq (i-1) \cdot 2^{-n+1}$.

By inductional assumption we have $C(k,j) \cap \mathrm{cl}\ C(g^{-1}((i-1) \cdot 2^{-n+1})) = \emptyset$ and $C(k,j) \cap \mathrm{cl}\ C(n,j) = \emptyset$ if $g(k,j) < (i-1) \cdot 2^{-n+1}$.

If $g(k,j) = (i-1) \cdot 2^{-n+1}$ then $C(n,i) \subseteq \mathrm{cl}\ C(k,j) - C(k,j)$ and $C(k,j) \cap \mathrm{cl}\,(C(n,i)) \subseteq C(k,j) \cap \mathrm{cl}\ (\mathrm{cl}(C(k,j)) - C(k,j))$.

Observe that the set $\mathrm{cl}(C(k,j)) - C(k,j)$ is closed and $C(k,j) \cap \mathrm{cl}(C(n,i)) = \emptyset$.

Let $k < n$ and $g(k,j) > g(n,i)$. Then $g(k,j) > (2i-1) \cdot 2^{-n}$ and $C(n,i) \subseteq X - \mathrm{cl}\,(C(k,j))$.

Since the set $C(n)$ is isolated, we have $C(n,i) \cap \mathrm{cl}\,(C(n,j)) = \emptyset$ for $i \neq j$.

Now we shall prove that (3) holds. Let $x \in C(n,m)$. By

(i) there is a neighbourhood U of x such that $U \cap \bigcup\{C(j,i): g(j,i) \geq g(n,m)$ & $j \leq n\} \subseteq \{x\}$.

Notice that $U \cap \bigcup\{C(j,i): g(j,i) \geq g(n,m)\} \subseteq \{x\}$. Indeed, suppose that there is $y \in C(j,i) \cap (U - \{x\})$ and $g(j,i) > g(n,m)$. By (ii) there is $t \geq m$ such that $y \in cl(C(n,t))$ and $(U - \{x\}) \cap C(n,t) \neq \emptyset$, a contradiction.

Thus, (3) holds.

PROPOSITION 7. Let $(X, \mathcal{T})$ be a T_1, dense in itself, Baire space. If C and C' are countable sums of isolated sets with $C \cap C' = \emptyset$ then there exists a function $f: X \to R$ such that $C = T(f)$ and $C' = T'(f)$.

Proof. By Lemma 3 we can assume that

$$C = \bigcup\{C(n,i) : n,i \in N, \quad i \leq 2^{n-1}\},$$

$$C' = \bigcup\{C'(n,i): n,i \in N, \quad i \leq 2^{n-1}\}$$

and the sets $C(n,i)$, $C'(n,i)$ have the properties (1)- (3).

Let $a = 2^{-1} \cdot 3^{-n}$ for $n = 1,2,\ldots$. Then $\sum_{n \in N} a_n = 4^{-1}$, $a_n = 2 \sum_{k > n} a_k$ and $a_n < 2^{-n-1}$.

Set

$$f(x) = \begin{cases} 2^{-n} & \text{for } x \in C(n,1), \\ f * C(g^{-1}((k-1) \cdot 2^{-n+1})) + a_n & \text{for } x \in C(n,k),\ 2 \leq k \leq 2^{n-1}, \\ 0 & \text{for } x \notin C \cup C', \\ -2^{-n} & \text{for } x \in C'(n,1), \\ f * C(g^{-1}((k-1) \cdot 2^{-n+1})) - a_n & \text{for } x \in C'(n,k),\ 2 \leq k \leq 2^{n-1}. \end{cases}$$

The sets $C(n,i)$ and $C'(n,i)$ are nowhere-dense and consequently, we have $\mathcal{T}\text{-}\limsup_{t \to x} f(t) \geq 0 \geq \mathcal{T}\text{-}\liminf_{t \to x} f(t)$ for each $x \in X$. Indeed, if $U \in \mathcal{T}$ and $x \in C(n,i) \cap U$ then there is a neighbourhood $V \subseteq U$ of x such that $C(n,i) \cap V = \{x\}$. Since X is dense in itself, we have $V - \{x\} \neq 0$. Since X is T_1, we have $V - \{x\} \in \mathcal{T}$.

Thus $V - \{x\}$ is an open, non-empty subset of U and $C(n,i) \cap (V - \{x\})$ is empty.

Let U be a neighbourhood of x. Since X is a Baire space, we have $U - C' \neq \emptyset$ and the set $\{t \in X: f(t) \geq 0\}$ is dense in X. Thus $\mathcal{T}\text{-}\limsup_{t \to x} f(t) \geq 0$. Similarly, the set $\{t \in X: f(t) \leq 0\}$ is dense and $\mathcal{T}\text{-}\liminf f(t) \leq 0$.

Let $x \in C(n,i)$, $\{b\} = f * C(g^{-1}((i-1)\cdot 2^{-n+1}))$ and $i > 1$. Then $f(x) = b + a_n$ and the set $U = X - cl(\cup\{C(j,k): g(j,k) > g(n,i)\} \cup (C(n) - \{x\}))$ is a neighbourhood of x such that $f(y) \leq b + \sum_{m>n} a_m = b + \frac{1}{2} a = f(x) - \frac{1}{2} a$ for each $y \in U - \{x\}$.

Thus $\mathcal{T}\text{-}\limsup_{t \to x} f(t) < f(x)$ and $x \in T(f)$.

Similarly. if $x \in C(n,1)$ then $f(y) \leq 2^{-n+1} + \sum_{m>n} a_m < 2^{-n} = f(x)$ for each $y \in U - \{x\}$ and consequently, $x \in T(f)$.

Thus $\bigcup_{n,i \in N} C(n,i) \subseteq T(f)$.

Similarly, we have $\bigcup_{n,i \in N} C'(n,i) \subseteq T'(f)$.

If $x \notin C \cup C'$ then $\mathcal{T}\text{-}\liminf_{t \to x} f(t) \leq 0 = f(x) \leq \mathcal{T}\text{-}\limsup_{t \to x} f(t)$.

Thus, $X - (C \cup C') \subseteq X - (T(f) \cup T'(f))$.

Let $X = R$. We have studied the sets $C(f)$, $S(f)$, $S'(f)$, $T(f)$ and $T'(f)$ with respect to several different topologies. For this reason we have adopted the convention of preceding each such denotation with the symbol for the topology. When no prefix appears it should be assumed that the Euclidean topology is meant.

For example, int A denotes the interior a set A in the Euclidean topology, $C(f)$ denotes the set of all points at which f is continuous with respect to the Euclidean topology and $C_{\mathcal{T}}(f)$ denotes the set of all points at which f is continuous with respect to $\mathcal{T}$.

"Measurable sets" denote subsets of R which are Lebesgue measurable. If A is measurable then $\varphi(A)$ denotes the set of all density points of A and m/M and m/A/ denotes the measure of A.

$\mathcal{J}$ denotes the σ-ideal of all meager sets. If A has the Baire property then $\Phi(A)$ denotes the set of all $\mathcal{J}$-density points of A [W].

We study the following topologies on R.

$\mathcal{T}_e$ - the Euclidean topology,

$\mathcal{T}_q$ - the qualitative topology (see [G]),

$\mathcal{T}_d$ - the density topology (see [GW], [O'M]),

$\mathcal{T}_r$ - the r topology (see [O'M])

$\mathcal{T}_{a.e.}$ - the a.e. topology (see [O'M])

$\mathcal{T}_I$ - the $\mathcal{J}$-density topology of Wilczyński (see [W])

$\mathcal{T}_L$ - the L-topology, $\mathcal{T}_L = \{G-A, \ G \in \mathcal{T}_s \text{ and } m(A) = 0\}$.

The relationships between $C(f)$, $C_{\mathcal{T}}(f)$, $S_{\mathcal{T}}(f)$, $S'_{\mathcal{T}}(f)$, $T_{\mathcal{T}}(f)$ and $T'_{\mathcal{T}}(f)$ are compared in Table 0. These and another connection between these sets are studied in [G],[N1],[N2],[N3],[N4].

REFERENCES

[E] Engelking R., Topologia ogólna, Warszawa 1976

[F] Fort jr. M.K., Category theorems, Fund. Math., 42 (1955), pp. 276-288.

[GW] Goffman C. and Waterman D., Approximately continuous transformations, Proc. Amer. Math. Soc., 12 (1961) pp. 116-121.

[G] Grande Z., Quelques remarques sur la semicontinuité supérieure, Fund. Math., 91, 1 (1976), pp. 1-13.

[Ł] Łojasiewicz St., Wstęp do teorii funkcji rzeczywistych, Warszawa 1973

[N1] Natkaniec T., On continuity and semicontinuity points, Real Anal. Exchange 9, No. 1 1983-84, pp. 215-232

[N2] Natkaniec T., On I-continuity and I-semicontinuity points, Math. Slovaca 36.3 (1986)

[N3] Natkaniec T., On points of the qualitative semicontinuity, Čas. pest. mat. 110 (1985) pp.337-342

[N4] Natkaniec T., Zbiory punktów ciągłych i półciągłych funkcji rzeczywistych, Ph. D. dissertation, Uniwersytet Łódzki, Łódź, 1984

[O'M] O'Malley R.J., Approximately differentiable functions the r topology, Pacific J. of Math., 72 (1977), pp.207-222

[O] Oxtoby J., Measure and category, Berlin-Heidelberg-New York 1971

[W] Poreda W., Wagner-Bojakowska E., and Wilczyński W., A category analogue of the density topology Fund. Math.

[S] Sierpiński W., Funkcje przedstawialne analitycznie, Warszawa 1925

[S1] Sikorski R., Funkcje rzeczywiste, t I, Warszawa 1958

Table 0

	$\mathcal{T}_e$	$\mathcal{T}_q$	$\mathcal{T}_d$	$\mathcal{T}_r$	$\mathcal{T}_{a.e.}$	$\mathcal{T}_I$	$\mathcal{T}_L$
$C_{\mathcal{T}}(f)$	G_δ set	$E-I$, E is G_δ and I is meager	a measurable set	$\bigcap_{m\in N}\bigcup_{n\in N} U_{mn} \cup J$, U_{mn} is an r-basis set, $m(J)=0$ and $D \subseteq \bigcap_{m\in N}\varphi(\bigcup_{n\in N} U_{mn})$	$\bigcap_{n\in N} H_n \cup J$, H_n is an open set, $m(J)=0$, $J\in\mathcal{J}$ and $J\in\bigcap_{n\in N}\varphi(H_n)$	$\bigcap_{n\in N} G_n \cup K$, G_n is an open set $K\in\mathcal{J}$ and $K\subseteq\bigcap_{n\in N}\Phi(G_n)$	a measurable set
$S_{\mathcal{T}}(f)$ $(S'_{\mathcal{T}}(f))$	every set	every set	every set	every set	every set	every set	every set
$T_{\mathcal{T}}(f)$ $(T'_{\mathcal{T}}(f))$	a countable set	a meager set	a measure zero set	a meager and measure zero set	$\bigcup_{n\in N} C_n$, $C_n\subseteq\varphi(R-\mathrm{cl}\,C_n)$ (thus $C_n\in\mathcal{J}$ and $m(C_n)=0$)	a meager set	a measure zero set
$S_{\mathcal{T}}(f)-C_{\mathcal{T}}(f)$ $(S'_{\mathcal{T}}(f)-C_{\mathcal{T}}(f))$	A: $\mathrm{int}A=\emptyset$	does not include second category sets with the Baire property	does not include measurable sets with positive measure	does not include r-basis sets	A: $\mathrm{int}A=\emptyset$	does not include second category sets with the Baire property	every set
$C_{\mathcal{T}}(f)-C(f)$	$\emptyset$	A: $\mathrm{int}A=\emptyset$ $(A\subseteq \mathrm{cl}(T_q(f)\cup T'_q(f)))$	?	?	$A=\bigcup_{n\in N} J_n$, $J_n\subseteq\varphi(R-\mathrm{cl}\,J_n)$ and $\mathrm{int}A=\emptyset$	?	A: $\mathrm{int}A\neq\emptyset$ $(A\subseteq\mathrm{cl}(T_L(f)\cup T'_L(f)))$

O PUNKTACH CIĄGŁOŚCI I PÓŁCIĄGŁOŚCI. PODSTAWOWE TWIERDZENIA

Streszczenie

W pracy badane są zbiory punktów ciągłości i półciągłości funkcji rzeczywistych $f:X \to R$ dla przestrzeni topologicznych $(X, \mathcal{T})$. Do pracy załączona jest tabela, w której porównuje się własności tych zbiorów gdy $X = R$ i $\mathcal{T}$ jest topologią euklidesową, topologią jakościową, topologią gęstości, r topologią, a.e. topologią oraz topologią I-gęstości Wilczyńskiego.