ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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HIGHER DEGREE FORMS AND m-APPLICATIONS[x)]

<u>1. Generalities</u>. We recall some definitions contained in [7], [1], and [2] . All rings and algebras will be commutative with 1.

<u>A polynomial law</u> on the pair (X,Y) of R-modules is a natural transformation $F=(F_A)\colon X\otimes - \longrightarrow Y\otimes -$, where $X\otimes -$, $Y\otimes -\colon$ R-Alg $\longrightarrow$ Set. It is called a <u>form of degree</u> <u>m</u> iff $F_A(\underline{x}a)=F_A(\underline{x})a^m$ for any A, $a\in A$ and $\underline{x}\in X\otimes A$. Then we denote $F\in\mathcal{P}^m_R(X,Y)$. It is proved in [7] that $\mathcal{P}^m_R(R^n,R)\simeq R[T_1,\dots,T_n]_m$. In the natural way we obtain the functor $\mathcal{P}^m_R\colon$ R-Mod^o x R-Mod $\longrightarrow$ R-Mod.

Any form $F\in\mathcal{P}^m_R(X,Y)$ has the following shape:

$$F_A(x_1\otimes a_1+\dots+x_n\otimes a_n)=\sum_{m_1+\dots+m_n=m} F_{m_1,\dots,m_n}(x_1,\dots,x_n)\otimes a_1^{m_1}\dots a_n^{m_n},$$

where $F_{m_1,\dots,m_n}\colon X^n\longrightarrow Y$ are uniquely determined by F. In particular $F_m=F_R$ and $F_{1,\dots,1}=PF$ is m-linear and symmetric. (We will assume that $m>0$). PF can be obtained from F_R in the following way:

$$PF(x_1,\dots,x_n)=(\Delta^m F_R)(x_1,\dots,x_m):=\sum_{i_1<\dots<i_k}(-1)^{m-k}F_R(x_{i_1}+\dots+x_{i_k})$$

Hence we have the natural transformation:

$$T^m_R\colon \mathcal{P}^m_R(X,Y)\longrightarrow \mathrm{Appl}^m_R(X,Y),\quad T^m_R(F)=F_R,$$

where $\mathrm{Appl}^m_R(X,Y)$ is the module of all <u>m-applications</u> $f\colon X\longrightarrow Y$, i.e. such mappings that $\Delta^m f$ is m-linear and $f(rx)=r^m f(x)$ for any $r\in R$ and $x\in X$. In the free case T^m_R gives us the following well-known mapping:

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x) Lecture given in Montpellier, October 1979, updated.

$T^m: R[T_1,\dots,T_n]_m \longrightarrow \mathrm{Appl}^m_R(R^n,R)$, $T^m(F)(x_1,\dots,x_n)=F(x_1,\dots,x_n)$

It is known that T^m_R is an isomorphism in the following cases:

(1) if $m\leq 2$ (2) if $m!\in U(R)$ (3) if $X=R$.

In the general case we will investigate $\mathrm{Ker}(T^m_R)=\tilde{\mathcal{P}}^m_R(X,Y)$ and $\mathrm{Im}(T^m_R)=\mathrm{Hom}^m_R(X,Y)\hookrightarrow \mathrm{Appl}^m_R(X,Y)$.

<u>2. Representability.</u> Observe that the functors in the following exact sequence:

$0 \longrightarrow \tilde{\mathcal{P}}^m(X,-) \hookrightarrow \mathcal{P}^m(X,-)\longrightarrow \mathrm{Appl}^m(X,-)$

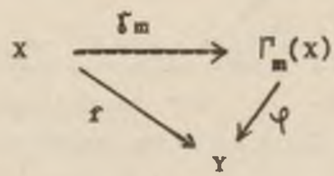
are representable.

(a) (see [7]) $\mathcal{P}^m(X,-)$ is represented by the m-th divided power $\Gamma^m(X)$ of X, and $F\in\mathcal{P}^m(X,Y)$ corresponds to $\varphi\in \mathrm{Hom}(\Gamma^m(X),Y)$ iff $F_{m_1,\dots,m_n}(x_1,\dots,x_n)=\varphi(x_1^{(m_1)}\dots x_n^{(m_n)})$.

(b) (see [1]) $\mathrm{Appl}^m(X,-)$ is represented by the module $\Gamma_m(X)$ defined by the generators $\gamma_m(x)$ for $x\in X$ and the following relations:

1° $\gamma_m(rx)=r^m\gamma_m(x)$ for any $r\in R$, $x\in X$ 2° $\Delta^m\gamma_m$ is m-linear.

The correspondence is given by the following diagram:



(c) (see [2]) T^m induces the natural homomorphism:

$h: \Gamma_m(X)\longrightarrow \Gamma^m(X)$, $h(\gamma_m(x))=x^{(m)}$.

Let $\bar{\Gamma}^m(X)=\mathrm{Im}(h)=R\{x^{(m)};\ x\in X\}$. Since Hom is left exact it follows that $\tilde{\mathcal{P}}^m(X,-)$ is represented by $\tilde{\Gamma}^m(X)=\mathrm{Coker}(h)=\Gamma^m(X)/\bar{\Gamma}^m(X)$.

(d) (see [2]) The functor $\mathrm{Hom}^m(X,-)$ is representable exactly in the case when the exact sequence $0\longrightarrow\bar{\Gamma}^m(X)\longrightarrow\Gamma^m(X)\longrightarrow\tilde{\Gamma}^m(X)\longrightarrow 0$ splits, and then it is represented by $\bar{\Gamma}^m(X)$. In the general case we have the exact sequence:

$0\longrightarrow \mathrm{Hom}^m(X,-)\hookrightarrow \mathrm{Hom}(\bar{\Gamma}^m(X),-)\longrightarrow \mathrm{Ext}^1(\tilde{\Gamma}^m(X),-)\longrightarrow$

$\rightarrow \mathrm{Ext}^1(\Gamma^m(X),-)$ (=0 if X is projective).

Example. It can be proved that $\Gamma^3_Z(Z^2)=Z^4$, $\tilde{\Gamma}^3_Z(Z^2)=Z_2$, and hence $\mathrm{Hom}^3_Z(Z^2,-)$ is not representable. The exact sequence is following:

$$0 \rightarrow \mathrm{Hom}^3_Z(Z^2,-) \rightarrow \mathrm{Hom}_Z(\Gamma^3_Z(Z^2),-) \rightarrow Z_2 \otimes - \rightarrow 0 .$$

3. The functor $\tilde{\Gamma}^m$. The fundamental properties of this functor, given in [2] and [3], are following:

(1) $\tilde{\Gamma}^m_R(X) \otimes A \simeq \tilde{\Gamma}^m_A(X \otimes A)$ for $A=R_S$, $A=R/I$ (but not in the general case !)

(2) Theorem. Let X be a finitely generated R-module. Then the following conditions are equivalent:

(i) $\tilde{\Gamma}^m_R(X)=0$

(ii) $\tilde{\Gamma}^m_{R/I}(X/IX)=0$ for any $I \in \mathrm{Max}(R)$

(iii) for any $I \in \mathrm{Max}(R)$ either $\dim_{R/I}(X/IX) \leq 1$ or $m \leq |R/I|$.

((i)$\Longleftrightarrow$(ii) is the Nakayama Lemma together with (1), (ii)$\Longleftrightarrow$(iii) is the case of a field).

Corollary. The following properties are equivalent :

(i) $\tilde{\Gamma}^m_R = 0$

(ii) $\tilde{\Gamma}^m_R(R^2)=0$

(iii) $m \leq d(R) := \inf\{|R/I| ;\ I \in \mathrm{Max}(R)\}$.

(3) Lemma . If $P \in \mathrm{Spec}(R)-\mathrm{Max}(R)$ then $\tilde{\Gamma}^m_R(X)_P=0$ for any R-module X. (R/P is infinite and hence $d(R_P)=\infty$. Next apply (1) and (2)).

Corollary. If $\dim(R)>0$ then $\tilde{\Gamma}^m_R(X)$ are torsion R-modules. (If non-zero, they are not free).

It is proved in [5] that $\vec{\Gamma}^m_R(X)$ is finite provided that R is noetherian and X is finitely generated.

(4) Structural theorems. Let R be a noetherian ring. Then:

(A) $\tilde{\Gamma}^m_R(X) \simeq \bigoplus_{P \in \mathrm{Max}(R)} \tilde{\Gamma}^m_{R_P}(X_P)$

(B) $\tilde{\Gamma}^m_R(X) \simeq \tilde{\Gamma}^m_{\hat{R}}(X \otimes \hat{R})$ if R is a local ring.

(C) Let X be a finitely generated R-module. Then

$$\tilde{\Gamma}^m_R(X) \simeq \bigoplus_{P \in \mathrm{Max}(R)} \tilde{\Gamma}^m_{R/P^{k(P)}}(X/P^{k(P)}X)$$

for all sufficiently large $k(P)$. If $m \leq 5$ (or $m \leq 7$ and $2 \in U(R)$) then we can choose $k(P)=1$ and hence:

$$\tilde{\Gamma}^m_R(X) \simeq \bigoplus_{P \in \mathrm{Max}(R)} \tilde{\Gamma}^m_{R/P}(X/PX)$$

for any R-module X. (Compare also [5], Corollary 5.10).

The above properties permit us to compute $Ker(T^m)$ in some cases. The question of Coker (T^m) is much more complicated.

4. Coker (T^m). Observe that $Hom^m(X,-) \subset Hom(\bar{\Gamma}^m(X),-) \subset Appl^m(X,-)$. If $Hom^m(X,-)$ is not representable (it is so very often), then the first two functors are different and hence $Coker(T^m)$ is non-zero. It was conjectured by M. Ferrero that the last two functors are equal, or, more precisely, that $\Gamma_m(X)$ and $\bar{\Gamma}^m(X)$ are isomorphic by h. This means that $Hom^m(X,Y) = Appl^m(X,Y)$ for any injective Y; in particular, over a field K, that $Coker(T^m_K)$ is zero for any K-modules. Unfortunately, it is not true in general.

Example. Let $K=F_4$, $X=K^4$, $Y=K$ and $f(x_1,x_2,x_3,x_4)=x_1^2x_2^2x_3^2x_4^2$. Observe that $f \in Appl^5_K(X,Y)$ because:

1° $f(rx)=r^8f(x)=r^5f(x)$ since $r^4=r$ in F_4,

2° $f=g^2$ where $g(x_1,x_2,x_3,x_4)=x_1x_2x_3x_4$ and hence $\Delta^5 f=(\Delta^5 g)^2=0$ since $()^2$ is additive in K and g is of degree 4.

On the other hand, f is „reduced" (all powers in f are less than $|K|$) and hence $f \notin Hom^m_K(X,Y)$ for $m < 8$. (see [4]).

$Coker(T^m)$ is completely described in [4] in the case of finite fields: any m-application is a polynomial mapping (but not necessarily of degree m - as in the example), and we can find the standard bases of Hom^m, $Appl^m$ and $Coker(T^m)$ for any K-modules. In particular, we have the following

Theorem. Let K be an algebraic extension of $\mathbb{Z}_p$ (for example a finite field of characteristic p). Then the following conditions are equivalent:

(1) $Coker(T^m_K)= 0$ (for any K-modules)

(2) $K=\mathbb{Z}_p$ or $m \leq 2p$.

In particular, $Coker(T^m_K)= 0$ for $m \leq 4$.

Remark 1. If $K \neq \mathbb{Z}_p$ and $Coker(T^m_K)= 0$ then $m \leq 2p \leq |K|$ and hence $Ker(T^m_K)=0$.

Remark 2. Using another methods we can prove that $Coker(T^m_K)=0$ over any field K for m=3, but the case of m=4 is unknown.

<u>Example.</u> Let K be an infinite algebraic extension of $\mathbb{Z}_2$. Then K is a sum of finite fields and we define $f\colon K^4 \longrightarrow K$ in the following way: if $x_1, x_2, x_3, x_4 \in K' \subset K$ and $|K'| = 2^s \geq 4$ then $f(x_1,x_2,x_3,x_4) = x_1^2 x_2^2 x_3^{2^{s-1}} x_4^{2^{s-1}}$.
As in the preceding example, it can be proved that f is a properly defined 5-application – but f is not a polynomial mapping.

Let us consider the homomorphism $h\colon \Gamma_m \dashrightarrow \bar{\Gamma}^m$ over rings. The problem of injectivity of h can be reduced to the local case, because Γ_m and $\bar{\Gamma}^m$ commute with localizations. Let (R,M) be a local ring. Then we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma_m(X)/M\,\Gamma_m(X) & \xrightarrow{\ \bar{h}\ } & \Gamma^m(X)/M\bar{\Gamma}^m(X) \\ \Big\downarrow f & & \Big\downarrow g \\ \Gamma_m(X/MX) & \xrightarrow{\ h'\ } & \bar{\Gamma}^m(X/MX) \end{array}$$

where the homomorphisms are defined in the natural way. They are all epi, but not iso in general (see [5]). Suppose that R/M is a proper algebraic extension of $\mathbb{Z}_p$ and $2p < m \leq |R/M|$. Then g is iso (since $\bar{\Gamma}^m = \Gamma^m$), h' is not iso for some free R-module X, and hence h is not iso. Unfortunately, the diagram doesn't give positive examples for our problem.

It is known from [6] that h is injective for $m=3$ (over R) in the following cases:

(1) for cyclic modules

(2) if R is von Neumann regular

(3) if no residue field of R is $\mathbb{Z}_2$

(4) if R is a DVR with the prime element 2

(5) if R is a Dedekind domain or $R = Z[w]$ (for flat R-modules).

<u>Example.</u> Let $m=3$. It follows from (3) and (4) that h is injective for $R=\mathbb{Z}$. The „generic'' 3-applications over $\mathbb{Z}$ are x^3, $xy(x \pm y)/2$ and xyz. If $R = \mathbb{Z}[T]$ then h is injec-

tive for flat, but not for all modules. If $R=S[X,Y]$ and (3) is not satisfied then h is not injective - even for the key case of R^2.

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FORMY WYŻSZYCH STOPNI I m-ODWZOROWANIA

Streszczenie

Praca stanowi tekst referatu wygłoszonego w Montpellier, poświęcona jest porównaniu dwóch definicji form wyższych stopni. W szczególności omówione zostały formy wytwarzające zerowe odwzorowania.