ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
Problemy Matematyczne 1983/84 z.5/6

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DERIVATIONS AND POLYNOMIAL RINGS

Let $d:R \longrightarrow R$ be a derivation, where R is a commutative ring with identity. Then d will be called locally nilpotent if for every $r \in R$ there exists n such that $d^n(r)= 0$.

This paper contains two parts. In the first part, we prove (Theorem 2.11) that if R has locally nilpotent derivations and satisfies additional assumptions, then R is a polynomial ring. We also prove some properties of locally nilpotent derivations. In the second part, we study the ring of constants for some derivations of a polynomial ring over an arbitrary field K.

We prove the following theorem (Theorem 3.21):

Let $R = K[x_1,x_2,\dots,x_n]$, where char $K = 0$. If $d:R \longrightarrow R$ is a derivation with $d(K)= 0$, $d \neq 0$, $d(x_i) \in K$, for $i=1,2,\dots,n$, then $C(R,d)$ is a polynomial ring (over K) in $n-1$ variables.

We also give other examples of rings of constants. At the end of this paper, we propose two problems.

1. <u>Preliminary notions.</u> All rings are commutative and have identity. A differential ring is a pair (R,d), where R is a ring and $d:R \rightarrow R$ is a mapping. called derivation, satisfying the conditions:

1) $d(a+b)= d(a)+ d(b)$, 2) $d(ab)= ad(b)+ d(a)b$, for arbitrary $a,b \in R$.

If d_1 and d_2 are derivations of R, then d_1+d_2 is a derivation of R.

If d is a derivation of R and $r \in R$, then rd is a derivation of R.

Immediately from the definition we get:

(1) $d(1)=0$

(2) $d(a^n)=na^{n-1}d(a)$

(3) $d^n(ab)=\sum_{k=0}^{n}\binom{n}{k}d^k(a)d^{n-k}(b)$

The formula (3) is known as Leibnitz formula (see [3]).

Let (R,d) be a differential ring and let $S=R[x_1,x_2,\dots,x_n]$ be a polynomial ring over R. The derivation d can be extended to S by setting for $d(x_1)$, $d(x_2)$, $d(x_3),\dots,d(x_n)$ arbitrary polynomials belonging to S (see [1]).

If $d:S\longrightarrow S$ is such derivation of S with $d(R)=0$, then for any $f\in S$, we have: (4) $d(f)=\sum_{k=1}^{n}\frac{\partial f}{\partial x_k}d(x_k)$ (see [1]).

If R contains the field Q of rational numbers, theh a differential ring (R,d) is called a Ritt algebra ([2]). We denote by $C(R,d)$ the set of all elements $r\in R$ such that $d(r)=0$. Then $C(R,d)$ is a subring of R called the ring of constants of the differential ring (R,d). If R is a field, then $C(R,d)$ is a subfield of R called the subfield of constants. If (R,d), (R_1,d_1) are differential rings, then a ring homomorphism $f:R\longrightarrow R_1$ (of rings with identity)is called a differential homomorphism if the diagram

$$\begin{array}{ccc} R & \xrightarrow{\ f\ } & R_1 \\ d\downarrow & & \downarrow d_1 \\ R & \xrightarrow{\ f\ } & R_1 \end{array}$$

is commutative.

If in addition we assume that $f:R\longrightarrow R_1$ is a ring isomorphism, then we say that f is a differential isomorphism.

LEMMA 1.1 Let (R,d), (R_1,d_1) be differential rings and let $f:R\longrightarrow R_1$ be a homomorphism of rings, not necessary a differential homomorphism. Then $A=\{r\in R,\ fd(r)=d_1f(r)\}$ is a subring of R.

PROOF: $1\in A$ since $fd(1)=f(0)=0=d_1(1)=d_1f(1)$. If $a,b\in A$, then $fd(a-b)=fd(a)-fd(b)=d_1f(a)-d_1f(b)=$ $=d_1f(a-b)$, so $a-b\in A$ and $fd(ab)=f(ad(b)+d(a)b)=$

$= f(a)f(d(b)) + f(d(a))f(b) = f(a)d_1(f(b)) + d_1(f(a))\cdot f(b) =$
$= d_1(f(a)f(b)) = d_1 f(ab)$, so $ab \in A$.

LEMMA 1.2. Let (R,d), (R_1,d_1) be differential rings and let $f: R \longrightarrow R_1$ be a ring homomorphism. Moreover, let R be an R_0-algebra and R is generated over R_0 by T. If $fd(a) = d_1 f(a)$, for $a \in R_0 \cup T$, then f is a homomorphism of differential rings.

PROOF: Let $A = \{r \in R,\ fd(r) = d_1 f(r)\}$. Since $R_0 \subseteq A$ and $T \subseteq A$, by Lemma 1.1 $R = R_0[T] \subseteq A$.

LEMMA 1.3. If $f:(R,d) \longrightarrow (R_1,d_1)$ is a differential isomorphism, then the rings $C(R,d)$ and $C(R_1,d_1)$ are isomorphio.

Proof is obvious.

__2. Locally nilpotent derivations.__ If (R,d) is a differential ring and n is natural, then by $C_n(R,d)$ we will denote the set $\{x \in R,\ d^n(x) = 0\}$

The sets $C_n(R,d)$ are subgroups of the additive group of R.

The following properties are obvious:

(1) $C_1(R,d) = C(R,d)$

(2) $C_1(R,d) \subseteq C_2(R,d) \subseteq C_3(R,d) \subseteq \dots$

THEOREM 2.1. Let (R,d) be a Ritt algebra without nilpotent elements. If $C_n(R,d) = R$, for some n, then $d = 0$

PROOF: Let $d^n = 0$ and $d^{n-1} \neq 0$, for some $n \geqslant 2$. Hence, there exists an element a of R such that $d^{n-1}(a) \neq 0$. Consider the derivation $\bar{d}: R[[t]] \longrightarrow R[[t]]$, ($R[[t]]$ is the formal power series ring) which is defined by the formula:
$\bar{d}\left(\sum_{k=0}^{\infty} r_k t^k\right) = \sum_{k=0}^{\infty} d(r_k)\, t^k$ and consider the automorphism $e: R[[t]] \longrightarrow R[[t]]$ defined by $e(a) = a + \frac{\bar{d}(a)}{1!}t + \frac{\bar{d}^2(a)}{2!}t^2 + \frac{\bar{d}^3(a)}{3!}t^3 + \dots\dots$ (see [3]).

For every $r \in R$, we have:

$$e(r) = r + \frac{d(r)}{1!}t + \frac{d^2(r)}{2!}t^2 + \dots + \frac{d^{n-1}(r)}{(n-1)!}t^{n-1}$$

Since $e(a^n) = e(a)^n$, we get

$$a^n + \frac{d(a^n)}{1!} t + \dots + \frac{d^{n-1}(a^n)}{} t^{n-1} =$$

$$= (a + \frac{d(a)}{1!} t + \frac{d^2(a)}{2!} t^2 + \dots + \frac{d^{n-1}(a)}{(n-1)!} t^{n-1})^n$$

If we compare coefficients at $t^{n(n-1)}$, we get $\left[\frac{d^{n-1}(a)}{(n-1)!}\right]^n = 0$. Hence, because R is Q-algebra without nilpotents, we have $d^{n-1}(a) = 0$

This contradicts to $d^{n-1}(a) \neq 0$.

For a differential ring (R,d) we define $E(R,d) = \bigcup_{n=1}^{\infty} C_n(R,d)$

PROPOSITION 2.2. $E(R,d)$ is a subring of R.

PROOF: Let $E = E(R,d)$. Since $d(1) = 0$, $1 \in E$. Let $x, y \in E$. We shall show that $x-y \in E$ and $xy \in E$. If $d^n(x) = 0$, $d^m(y) = 0$ and $k = \max(n,m)$ then $d^k(x-y) = d^k(x) - d^k(y) = 0$, so $x - y \in E$. Further, $d^{n+m}(xy) = \sum_{i=0}^{n+m} \binom{n+m}{i} d^i(x) d^{n+m-i}(y) = 0$, because for $i = 0, 1, \dots, n$, $d^{n+m-i}(y) = 0$, and for $i = n+1, n+2, \dots, n+m$, $d^i(x) = 0$. Finally $xy \in E$.

LEMMA 2.3. If there exists n such that $C_n(R,d) = C_{n+1}(R,d)$, then $E(R,d) = C_n(R,d)$.

PROOF: Let $C_m = C_m(R,d)$, for $m = 1, 2, \dots$. We prove (by induction on s) that for any natural s we have $C_{n+s} = C_n$.

Let $x \in C_{n+s}$. Then $0 = d^{n+s}(x) = d^{n+s-1}(d(x))$, so $d(x) \in C_{n+s-1}$ and by induction $d(x) \in C_n$. Hence $0 = d^n(d(x)) = d^{n+1}(x)$, and $x \in C_{n+1} = C_n$

DEFINITION 2.4. A derivation $d: R \longrightarrow R$ is said to be locally nilpotent iff $E(R,d) = R$. In other words, a derivation d is locally nilpotent iff for any $r \in R$, there exists n such that $d^n(r) = 0$

EXAMPLE 2.5. Let $C[x]$ be a polynomial ring in one variable x, with coefficients in a ring C. If $d: C[x] \longrightarrow C[x]$ is such derivation that $d(C) = 0$, $d(x) = 1$, then d is locally nilpotent. Indeed, for $f \in C[x]$ we have $d^{n+1}(f) = 0$ where $n = \deg f$.

In some cases, we can prove the theorem inverse to the result given in Example 2.5.

THEOREM 2.6. Let d be a locally nilpotent derivation of a Q-algebra R. If there exists an element $x \in R$ such that $d(x)= 1$, then R is isomorphic to the polynomial ring $C[t]$, where $C = C(R,d)$. Precisely, there exists a differential isomorphism $\varphi :(C[t] , \frac{\partial}{\partial t}) \longrightarrow (R,d)$.

PROOF: Let $C = C(R,d)$ and let $\varphi : C[t] \longrightarrow R$ be a ring homomorphism such that $\varphi(t)= x$, $\varphi|_C = 1_C$. Since $d\varphi(t)= d(x)= 1 = \frac{\partial}{\partial t}(t)= \varphi\frac{\partial}{\partial t}(t)$, by Lemma 1.2, we have that φ is a differential homomorphism from $(C[t], \frac{\partial}{\partial t})$ to (R,d). We show that φ is injective. Let $w = c_n t^n + \ldots + c_1 t + c_o$, be an element of $C[t]$ with $\varphi(w)= 0$.
We get $0 = \varphi(c_n t^n + \ldots + c_o)= c_n x^n + \ldots + c_o$, and next
$0 = d^n(c_n x^n + \ldots + c_o)= n!c_n$.
Since R is a Q-algebra, the equality $n!c_n = 0$ implies $c_n = 0$. In a similar manner, we get $c_n = c_{n-1}= \ldots = c_1 = c_o$, hence $w = 0$. Now we show that φ is surjective. Let $r \in R$. Since $R = E(R,d)$, there is n such that $r \in C_n(R,d)$. By induction on n we prove that $r \in \operatorname{Im}\varphi$.

a) If $n = 1$, then $d(r)= 0$, and $r = \varphi(r)$.

b) Suppose $d^n(r)= 0$. Then $d(r) \in C_{n-1}(R,d)$, and by induction $d(r)= \varphi(w)$, where $w \in C[t]$.
Suppose $w = c_k t^k + \ldots + c_1 t + c_o$. Put $u= \frac{c_k}{k+1} \cdot t^{k+1} + \frac{c}{k}{}_{k-1} \cdot t^k + \ldots \frac{c}{1}{}_o \cdot t$. Then $u \in C[t]$ and $\frac{\partial}{\partial t} u = w$.
Since $d(r - \varphi(u))= d(r)- d(\varphi(u))= \varphi(w)- \varphi\frac{\partial}{\partial t}(u)= \varphi(w)- \varphi(w)= 0$, we get $r - \varphi(u)= c \in C$, and hence $r - \varphi(u)=\varphi(c)$.
Finally $r = \varphi(u + c)$, where $u + c \in C[t]$, that means $r \in \operatorname{Im}\varphi$.

REMARK. A different proof can be found (but for domains) in [4] . Now we give some generalizations of Theorem 2.6.

COROLLARY 2.7. Let $d_1, d_2, \ldots, d_n$ be commutative locally nilpotent derivations in a Q-algebra R. If in R there exist elements $x_1, x_2, \ldots, x_n$ such that
$d_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$, then R is isomorphic to the

polynomial ring $C[t_1, t_2, \ldots, t_n]$, where
$C = \{r \in R,\ 0 = d_1(r) = d_2(r) = \ldots = d_n(r)\}$.

PROOF. Induction on n. For $n = 1$, the corollary coincides with Theorem 2.6. Suppose $n > 1$, and put
$\bar{C} = \{r \in R,\ d_2(r) = \ldots = d_n(r) = 0\}$. Now by induction
$R = \bar{C}[t_2, \ldots, t_n]$. Notice that $d_1(\bar{C}) \subset \bar{C}$
Indeed, if $r \in \bar{C}$, then for $i=1,2,\ldots,n$ we get
$d_i d_1(r) = d_1 d_i(r) = d_1(0) = 0$. Therefore d_1 is a (locally nilpotent) derivation in the Q-algebra $\bar{C}$. Since $x_1 \in \bar{C}$ and $d_1(x_1) = 1$, by Theorem 2.6, it follows that $\bar{C} \approx \bar{\bar{C}}[t_1]$ where $\bar{\bar{C}} = \{a \in \bar{C},\ d_1(a) = 0\}$. Notice that $\bar{\bar{C}} = C$.
Finally we have : $R \approx \bar{C}[t_2, \ldots, t_n] \approx (\bar{\bar{C}}[t_1])[t_2, \ldots, t_n] \approx \bar{\bar{C}}[t_1, \ldots, t_n]$ $= C[t_1, \ldots, t_n]$.
The following three lemmas will be needed in further generalizations of Theorem 2.6.

LEMMA 2.8. Let d_1 and d_2 be commutative derivations in a ring R. If d_1 and d_2 are locally nilpotent, then $d_1 + d_2$ is a locally nilpotent derivation too.

PROOF. If $d_1 d_2 = d_2 d_1$, then $(d_1 + d_2)^n = \sum_{k=0}^{n} \binom{n}{k} d_1^k d_2^{n-k}$.
Now if for $r \in R$ we have $d_1^n(r) = 0$, $d_2^m(r) = 0$, then $(d_1 + d_2)^{n+m}(r) = 0$.

LEMMA 2.9. If d is a locally nilpotent derivation of R and $a \in C(R,d)$, then ad is a locally nilpotent derivation of R.

PROOF. It follows from the formula : $(ad)^n(x) = a^n d^n(x)$.

LEMMA 2.10. Let $d_1, d_2, \ldots, d_n$ be derivations in a ring R, and let $x_1, x_2, \ldots, x_n$ be such elements of R that the matrix $[d_i(x_j)]$ is invertible. Then there exist derivations $\delta_1, \delta_2, \ldots, \delta_n$ of R with $\delta_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

PROOF. Let $[a_{ij}]$ be the matrix over R such that $[a_{ij}][d_{ij}(x)] = I$, where I is the identity. Take
$\delta_i = a_{i1}d_1 + a_{i2}d_2 + \ldots + a_{in}d_n$, for $i=1,2,\ldots,n$. Then $\delta_1, \delta_2, \ldots, \delta_n$ are derivations of R such that
$\delta_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$.

THEOREM 2.11. Let $d_1, d_2, \ldots, d_n$ be commutative locally nilpotent derivations of a Q-algebra R and let $x_1, x_2, x_3, \ldots, x_n$ be such elements in R that the matrix $A = [d_i(x_j)]$ is invertible. Moreover, let
$C = \{r \in R,\ 0 = d_1(r) = d_2(r) = \ldots = d_n(r)\}$.
If A^{-1} is the matrix with coefficients in C, then the ring R is isomorphic to a polynomial ring in n variables over a subring R_o.

PROOF. By Lemma 2.10, there exist derivations $\delta_1, \delta_2, \ldots, \delta_n$ such that $\delta_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$. By the construction of these derivations (see the proof of Lemma 2.10), and by Lemmas 2.8, 2.9 it follows that $\delta_1, \delta_2, \ldots, \delta_n$ are commutative locally nilpotent derivations.
The result follows from Corollary 2.7.
At the end of this part we note some properties of locally nilpotent derivations.

THEOREM 2.12. Let (R,d) be a Ritt algebra without nilpotent elements. If d is a nonzero derivation which is locally nilpotent, then $C_1(R,d) \subsetneq C_2(R,d) \subsetneq C_3(R,d) \subsetneq \ldots$.

PROOF. Suppose that for some integer n, $C_n(R,d) = C_{n+1}(R,d)$. Then, by Lemma 2.3, we have $R = E(R,d) = C_n(R,d)$. Whence, by Theorem 2.1, we get $d = 0$.

3. The ring of constants for some derivations in $K[x_1, x_2, \ldots, x_n]$

Now we shall describe the ring $C(R,d)$ in the case $R = K[x_1, \ldots, x_n]$, where K is a field of characteristic zero, and d is a derivation of R with $d(K) = 0$, $d(x_i) \in K$, for $i = 1, 2, \ldots, n$.

LEMMA 3.1. If R is a Q-algebra without zero divisors, and $d: R[x] \longrightarrow R[x]$ is a derivation with $d(R) = 0$, $d(x) \neq 0$, then $C(R[x], d) = R$.

PROOF. Let $C = C(R[x], d)$. Evidently $R \subseteq C$. Suppose $f \in C \setminus R$. Then $\deg f = n \geqslant 1$. If $f = a_n x^n + \ldots + a_o$, where $a_n \neq 0$, then $0 = d(f) = (na_n x^{n-1} + \ldots + a_o)d(x)$. Hence it follows $na_n = 0$, and $a_n = 0$. This contradiction proves the Lemma.

THEOREM 3.2. Let K be a field of characteristic zero, and let $R = K[x_1,\dots,x_n]$. If $d:R \longrightarrow R$ is a derivation such that :

a) $d(K)= 0$

b) $d \neq 0$

c) $d(x_i)\in K$, for $i=1,2,\dots,n$

then $C(R,d)$ is a polynomial ring over K in $n-1$ variables.

PROOF. The case 1. Let $d(x_1)= 1$, $d(x_2)= \dots = d(x_n)=0$. Let $S = K[x_2,x_3,\dots,x_n]$. Then $R = S[x_1]$, $d(S)= 0$ and by Lemma 3.1 we have $C(R,d)= S = K[x_2,x_3,\dots,x_n]$.

The case 2. Let $d(x_1)= \dots = d(x_s)= 1$, $d(x_{s+1})= \dots = d(x_n)= 0$ where $1 \le s \le n$. Consider a ring isomorphism $\varphi: K[y_1,\dots,y_n] \longrightarrow K[x_1,\dots,x_n]$ such that :

$$\varphi(y_1)= x_1$$
$$\varphi(y_2)= x_1 - x_2$$
$$\vdots$$
$$\varphi(y_s)= x_1 - x_s$$
$$\varphi(y_{s+1})= x_{s+1}$$
$$\vdots$$
$$\varphi(y_n)= x_n$$

Let $\delta: K[y_1,\dots,y_n] \longrightarrow K[y_1,\dots,y_n]$ be a derivation such that $\delta(K)=0$, $\delta(y_1)= 1$, $\delta(y_2)= \dots = \delta(y_n)= 0$.
It is easy to verify that $d\varphi(y_i)=\varphi\delta(y_i)$, for $i=1,2,\dots,n$.
By Lemma 1.2 φ is a differential isomorphism. Therefore $C(R,d)$ is isomorphic to $C(K[y_1,\dots,y_n],\delta)$, and from the case 1 we get that $C(R,d)$ is isomorphic to $K[y_2,\dots,y_n]$.

The case 3. (the general situation). Since $d\neq 0$, there is $i\in\{1,2,\dots,n\}$ such that $d(x_i)\neq 0$.
Let $d(x_i)= a_i$. Without loss of generality we can assume that $a_i \neq 0$, for $i=1,2,\dots,s$ and $a_i = 0$, for $i=s+1,\dots,n$.
Consider a ring isomorphism $\varphi : K[y_1,\dots,y_n] \longrightarrow K[x_1,\dots,x_n]$ which is defined by the following formulas :

$$\varphi(y_i) = a_i^{-1} x_i \ , \quad \text{if} \quad i=1,2,\dots,s$$
$$\varphi(y_j) = x_j \ , \quad \text{if} \quad j=s+1,\dots,n$$

Let δ be a derivation in $K[y_1,\dots,y_n]$ such that $\delta(K)=0$, $\delta(y_1)= \dots = \delta(y_s)= 1$ and $\delta(y_{s+1})= \dots = \delta(y_n)= 0$.
Notice that $d\varphi(y_i)= \varphi\delta(y_i)$, for $i=1,\dots,n$.
Indeed, if $i=1,2,\dots,s$, then $d\varphi(y_i)= d(a_i^{-1}x_i)=a_i^{-1}d(x_i)= = a_i^{-1}a_i = 1$

$\varphi\delta(y_i)=\varphi(1)= 1$

if $j=s+1,\dots,n$, then $d\varphi(y_j)= d(x_j)= a_j= 0$

$\varphi\delta(y_j)= \varphi(0)= 0.$

By Lemma 1.2 φ is a differential isomorphism. Thus by Lemma 1.3, we get $C(R,d) \simeq C(K[y_1,\dots,y_n],\delta)$, and the case 2, we get that $C(R,d)$ is isomorphic to $K[y_2,\dots,y_n]$. This ends the proof.

Now we give some remarks on the $C(R,d)$ in the case $R = K[x_1,\dots,x_n]$ and if d is a derivation of R such that $d(K)= 0$ and $d(x_i)\notin K$, for some i.

PROPOSITION 3.3. Let $R = K[x_1,\dots,x_n]$ be a polynomial ring over a field K of characteristic zero. If $d:R \longrightarrow R$ is a derivation such that $d(K)= 0$ and $d(x_i)= x_i$, for $i=1,2,\dots,n$, then $C(R,d)= K$.

PROOF. If $u=x_1^{i_1}\dots x_n^{i_n}$, where $i_1,i_2,\dots,i_n \geqslant 0$, then $d(u)= (i_1+ \dots i_n)u$

Assume that $F = \sum k_{i_1\dots i_n} x_1^{i_1} \dots x_n^{i_n}$ (where $k_{i_1\dots i_n} \neq 0$), is a polynomial belonging to $C(R,d)$.
Then $0 = d(F)=\sum(i_1 + \dots + i_n)k_{i_1\dots i_n} x_1^{i_1} \dots x_n^{i_n}$, and hence $i_1 + \dots + i_n = 0$, that means $i_1 = i_2 = \dots = i_n = 0$. and finally $F\in K$.

PROPOSITION 3.4. Let $R = K[x_1,x_2,\dots,x_n]$ be a polynomial ring over a field K of characteristic $p>0$ and let $d:R \longrightarrow R$ be a derivation with $d(K)= 0$, $d(x_i)= x_i$, for $i=1,2,\dots,n$.
Let $F =\sum k_{i_1\dots i_n} x_1^{i_1} \dots x_n^{i_n}$ be an element of R. Then $F\in C(R,d)$ if and only if $p \mid (i_1 + \dots + i_n)$, for all sequences $(i_1,\dots,i_n)$ such that $k_{i_1\dots i_n} \neq 0$.

PROOF. Analogous to the proof of Proposition 3.3.

PROPOSITION 3.5. Let $R = K[x,y]$, char $K = 0$ and let $d: R \longrightarrow R$ be a nonzero derivation such that $d(K) = 0$, and

$$d(x) = Ax + By$$
$$d(y) = Cx + Dy, \text{ where } A, B, C, D \in K.$$

If $AD - BC = 0$, then the ring $C(R,d)$ is isomorphic to the polynomial ring over K in one variable.

PROOF. Since $d \neq 0$, one of A, B, C, D is not zero. Suppose $A \neq 0$. Then $Cx + Dy = A^{-1}(ACx + ADy) = A^{-1}(ACx + BCy) = A^{-1}C(Ax + By)$.

Now we have $d(x) = Ax + By$

$d(y) = k(Ax + By)$, where $Ax + By \neq 0$, $k \in K$.

Let $\delta: R \longrightarrow R$ be a derivation such that $\delta(K) = 0$, $\delta(x) = 1$, $\delta(y) = k$. Then $C(R,d) = C(R, (Ax + By)\delta) = C(R, \delta) \approx K[t]$, where the last isomorphism we get by Theorem 3.2.

A description of a ring of constants for derivations in a polynomial ring is a difficult problem. It is complicated, even in the oase of two variables. Consider the examples below:

EXAMPLE 3.6. Let $R = K[x,y]$, char $K = 0$ and $d: R \rightarrow R$ be a derivation

1) If $d(x) = y$, $d(y) = -x$, then $C(R,d) = K[x^2 + y^2]$.
2) If $d(x) = x + y$, $d(y) = x$, then $C(R,d) = K$.
3) If $d(x) = x + n(n + 1)y$, $d(y) = x$, then the element $(x - (n+1)y)^{n+1}(x+ny)^n$ belongs to $C(R,d)$
4) If $d(x) = xy$, $d(y) = -x^2 - y^2$, then $x^4 + 2x^2y^2 \in C(R,d)$.
5) If $d(x) = 3x^2y - 1$, $d(y) = -4xy^2$, then $y(x^2y - 1)^2 \in C(R,d)$.

Now we propose two problems.

PROBLEM 1. Let $R = K[x,y]$, char $K = 0$ and let $d: R \rightarrow R$ be such a derivation that $d(K) = 0$,

$d(x) = Ax + By$

$d(y) = Cx + Dy$, where $A, B, C, D \in K$.

a) Is $C(R,d)$ a Noetherian ring ?
b) Is $C(R,d)$ a finitely generated K-algebra ?
c) Find necessary and sufficient conditions for $C(R,d) = K$.
d) Find necessary and sufficient conditions for $C(R,d) \approx K[t]$.

PROBLEM 2. Let d be a nonzero derivation in $K[x,y]$ such that $d(K)=0$. Is there a situation for which $C(R,d)\neq K$ and $C(R,d)\not\supset K[t]$?

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DERYWACJE A PIERŚCIENIE WIELOMIANÓW

Streszczenie

Derywację $d:R\to R$ nazywamy lokalnie nilpotentną wtedy i tylko wtedy, gdy $d^n(r)=0$, dla pewnego $n\in N$ oraz dla każdego $r\in R$ (R - pierścień przemienny). W pracy rozpatrujemy szereg własności derywacji lokalnie nilpotentnych. Dowodzimy między innymi (Tw. 2.11), że jeżeli R posiada derywacje lokalnie nilpotentne oraz spełnia dodatkowe założenia, to R jest pierścieniem wielomianów nad pewnym ciałem K.

W drugiej części pracy zajmujemy się badaniem pierścieni stałych dla pewnych derywacji pierścieni wielomianów nad ciałem. Artykuł kończymy sformułowaniem dwóch otwartych problemów.