ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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SOME REMARKS ON SYSTEMS OF IDEALS (II)

Systems of ideals in commutative rings have been investigated in [4],[5]. Recall that a pair (R,M) is said to be a system of ideals, if R is a commutative ring and M is a set of ideals of R satisfying the following conditions:

A1. R is an element of M,

A2. The intersection of any set of elements of M is an element of M,

A3. The union of any non-empty set of elements of M, totally ordered by inclusion, is an element of M,

A4. The null ideal belongs to M,

A5. If A,B belong to M, then A+B belongs to M,

A6. If A,B belong to M, then AB belongs to M,

A7. If A,B belong to M, then (A:B)belongs to M,

A8. If A belongs to M, and x is any element of R, then $A_x = \bigcup_{n=0}^{\infty} (A:x^n)$ belongs to M.

For any system of ideals (R,M) in R we have two natural operations $\#$: I(R)$\longrightarrow$ I(R) and [ ] : I(R)$\longrightarrow$ I (R) on the set I(R) of all ideals of R such that, for A from I(R), $A_{\#}$ is the greatest M-ideal (an ideal from M) contained in A and A is the smallest M-ideal containing A. These operations are useful tools in the proofs of many theorems in the theory of differential rings ([1],[2],[3]) and in general theory of systems of ideals ([4],[5]).

In this note we define axiommaticaly two kinds of operations on ideals of rings, called the interior and closing operations, and show that there is a one-one correspondence between the set of all interior operations (resp. closing operations) of a fixed ring R and the set of ideal systems in R.

DEFINITION 1. A mapping $\alpha : I(R) \longrightarrow I(R)$ is said to be an <u>interior operation</u> on ideals of R iff it satisfies the following conditions

W1. $\alpha(A) \subset A$

W2. $\alpha(\alpha(A)) = \alpha(A)$

W3. $\alpha(\bigcap A_i) = \bigcap \alpha(A_i)$

W4. $\alpha(R) = R$

W5. $\alpha(AB) \supset \alpha(A)\alpha(B)$

W6. $\alpha(A : \alpha(B)) = (\alpha(A) : \alpha(B))$

W7. $\alpha(\alpha(A)_x) = \alpha(A)_x$, for every $x \in R$.

DEFINITION 2. A mapping $\gamma : I(R) \longrightarrow I(R)$ is said to be a <u>closing operation</u> on ideals of R iff it satisfies the following conditions

D1. $A \subset \gamma(A)$

D2. $\gamma(\gamma(A)) = \gamma(A)$

D3. If $\{A_i\}_{i \in I}$ is a subset of $I(R)$ totally ordered by the inclusion, then $\gamma(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} \gamma(A_i)$

D4. $\gamma(0) = 0$

D5. $\gamma(A+B) = \gamma(A) + \gamma(B)$

D6. $\gamma(AB) \subset \gamma(A)\gamma(B)$

D7. $\gamma(A : \gamma(B)) \subset (\gamma(A) : \gamma(B))$

D8. $\gamma(\gamma(A)_x) = \gamma(A)_x$

LEMMA 1. If (R,M) is a system of ideals, then $\#$ is an interior operation, and $[\ ]$ is a closing operation on ideals of R.

PROOF. Most of the conditions W1-W7, D1-D8 follows from the definitions . We verify the condition W6 . First we check the ionlusion $(A:B)_{\#} \subset (A_{\#} : B_{\#})$. Since $B(A:B) \subset A$ and, by the condition W5, $B_{\#}(A:B)_{\#} \subset (B(A:B))_{\#}$, we have $(A:B)_{\#} \subset (A_{\#} : B_{\#})$. Hence, since $B_{\#} \in M$, $(A:B_{\#})_{\#} \subset (A_{\#} : ((B)_{\#})_{\#}) = (A_{\#} : B_{\#})$. Conversely, the inclusion $(A_{\#} : B_{\#}) \subset (A : B_{\#})$ gives

$(A_{\#}:B_{\#})_{\#} \subset (A:B_{\#})_{\#}$. Finally, by A7, $(A_{\#}:B_{\#}) \in M$ and consequently $(A_{\#}:B_{\#})_{\#} = (A_{\#}:B_{\#})$ and $(A_{\#}:B_{\#}) \subset (A:B_{\#})_{\#}$.

THEOREM 1. Let R be a commutative ring with identity.

a) There is a bijection between the set of all interior operations on ideals of R, and the set of all $M \subset I(R)$ such that (R,M) is a system of ideals.

b) There is a bijection between the set of all closing operations on ideals of R, and the set of all $M \subset I(R)$ such that (R,M) is a system of ideals.

PROOF. Proofs of a) and b) are similar, so we prove only a). Let α be an interior operation on ideals of R. Let $M_{\alpha} = \{A \in I(R),\ \alpha(A) = A\}$. We shall verify that M_{α} satisfies the conditions A1-A8. The conditions A1, A2, A4, A7, A8 are obvious. We check the remaining conditions. A3. Let $\{A_i\}_{i \in I}$ be a subset of M, totally ordered by inclusion. Then by W1 and W3 $\alpha(\bigcup A_i) \subset \bigcup A_i = \bigcup \alpha(A_i) \subset \alpha(\bigcup A_i)$, and $\bigcup A_i \in M_{\alpha}$. A5, A6. If $A,B \in M_{\alpha}$, then, applying W1 and W3 again, we get $\alpha(A+B) \subset A+B = \alpha(A)+\alpha(B) \subset \alpha(A+B)$ and $\alpha(AB) \subset AB = \alpha(A)\alpha(B) \subset \alpha(AB)$. Thus A+B, AB belong to M_{α}. By Lemma 1, we know that every system of ideals in R has the form M_{α}. Indeed, if (R,M) is a system, then the operation $\# : I(R) \to I(R)$ defined by this system is such interior operation on ideals of R that $M = M_{\#}$. It remains to show that $M_{\alpha} = M_{\beta}$, for operations α, β, implies $\alpha = \beta$. Let $A \in I(R)$. Then $\alpha(A) \in M_{\alpha} = M_{\beta}$, $\beta(A) \in M_{\beta} = M_{\alpha}$, and consequently $\beta(\alpha(A)) = \alpha(A)$ and $\alpha\beta(A) = \beta(A)$. Hence by W3 $\alpha(A) = \beta\alpha(A) \subset \beta(A)$ and $\beta(A) = \alpha\beta(A) \subset \alpha(A)$, that means $\alpha = \beta$.

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PEWNE UWAGI DOTYCZACE SYSTEMOW IDEAŁÓW (II)

Streszczenie

W teorii pierścieni różniczkowych jak również w teorii systemów ideałów ważną rolę odgrywają dwie operacje #, [ ]: I(R)⟶ I (R) zadane na zbiorze I(R)wszystkich ideałów danego pierścienia R (patrz [1],[2],[3]). W niniejszej pracy wprowadzamy aksjomatycznie dwa rodzaje operacji na ideałach pierścienia R, nazywane operacjami wnętrza i domknięcia. Dowodzimy twierdzenie, które głosi, że operacje te tworzą zbiory, które są izomorficzne z rodziną wszystkich systemów ideałów pierścienia R.