ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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SOME RESULTS ON INTERPRETABILITY WITH PARAMETERS (abstract)

The concept of interpretability of theories was introduced in "Undecidable Theories" [1] . From this time, there have arised a lot of different formalization of this notion, though the relation of interpretability is always a quasi-order on the some class of theories. Thus one can concider the equivalence relation (mutual interpretability)defined in usual way by this quasi-order and then the partial-order on classes of equivalence, which is induced by interpretability.

Mycielski showed [2] , that in case of his definition of interpretability ("local interpretability") this induced partial-order is of the lattice type. He also proved, that this lattice is distributive, complete and algebraic.

Lindström showed [3] , that in case of Tarski's definition of interpretability for every theory T - a consistent recursive extention of Peano's arithmetic, the set of equivalence classes of theories, which are the extentions of T, is a distributive lattice with minimal and maximal elements. This lattice is complemented if the theory T is not Σ_1-correct.

In this paper I shall give (without the proofs)some similar results concerning interpretability with parameters, which was introduced by Szczerba ([4],[5]).

We consider languages of finite signature only. By a theory we always mean a consistent set of sentences in some language of finite signature. Theory is always closed under deduction.

We shall use the following notation:

L_σ - the language of signature σ ;

St_σ - the class of all structures of signature σ ;

L_σ^n - the set of these formulas of L_σ in which at most the variables $x_0, x_1, \ldots, x_{n-1}$ occur free;

$\varphi^{\mathfrak{A},\bar{a}} = \{\bar{b} \in A^n : \mathfrak{A} \models \varphi[\bar{a} \frown \bar{b}]\}$ where $\mathfrak{A} \in St_\sigma$, $\varphi \in L^{m+n}$, $\bar{a} \in A^m$.

Let us now give some basic definitions.

Definition 1

Let σ, τ be two signatures.

By a code (with parameters) from σ to τ we mean the sequence $c = \langle n, \varphi_p, \varphi_u, \varphi_=, \varphi_0, \varphi_1, \ldots, \varphi_{lh\tau-1}$ where n is an arbitrary natural number, $\varphi_p \in L^m$, $\varphi_u \in L_\sigma^{m+n}$, $\varphi_= \in L_\sigma^{m+2n}$, $\varphi_i \in L^{m+\tau_i n}$ for $i < lh\tau$.

Definition 2

Let c be a code from σ to τ.

Let $\mathfrak{A}$ be a structure of signature σ with the universe A.

For each $\bar{a} \in A^m$, if $\varphi_u^{\mathfrak{A},\bar{a}}$ is nonempty and $\varphi_=^{\mathfrak{A},\bar{a}}$ is a congruence relation in $\mathfrak{A}_c^{\bar{a}} = (A^n, (\varphi_i^{\mathfrak{A},\bar{a}})_{i<lh\tau}) \upharpoonright \varphi_u^{\mathfrak{A},\bar{a}}$

then we define $\Gamma c(\mathfrak{A},\bar{a}) = \mathfrak{A}_c^{\bar{a}} / \varphi_=^{\mathfrak{A},\bar{a}}$.

Now, let us define

$\Gamma c\, \mathfrak{A} = \{\Gamma c(\mathfrak{A},\bar{a}) : \mathfrak{A} \models \varphi_p[\bar{a}]\}$.

Definition 3

Let c be a code from σ to τ.

Let $\bar{x}$, $\bar{x}_i$ denote the sequence of variables $(x_0, x_1, \ldots x_{m-1})$, $(x_{m+in}, x_{m+in+1}, \ldots, x_{m+in+n-1})$ respectively.

Let us inductively define the function $Fc : L_\tau \longrightarrow L_\sigma$ in the following way:

$Fc\,(x_i = x_j) = \varphi_u(\bar{x}_i) \,\&\, \varphi_u(\bar{x}_j) \,\&\, \varphi_=(x_i, x_j)$;

$Fc\,(R_i(x_{j_0}, x_{j_1}, \ldots, x_{j_{\tau_i-1}})) = \bigwedge_{k=0}^{\tau_i-1} \varphi_u(\bar{x}_{j_k}) \,\&\, \varphi_i(\bar{x}_{j_0}, \ldots, \bar{x}_{j_{\tau_i-1}})$

where R_i is τ_i-ary symbol of L_τ;

$Fc\,(\neg\varphi) = \neg\, Fc\,(\varphi)$;

$Fc(\varphi \vee \psi) = Fc\,(\varphi) \vee Fc\,(\psi)$;

$Fc\,(\forall x_i \varphi) = \forall \bar{x}_i\,(\varphi_u(\bar{x}_i) \longrightarrow Fc(\varphi))$.

Definition 4

Let T_1 be the theory in L_τ and T_2 be the theory in L_σ.
Then (a) Let c be a code from σ to τ.

We say, that c interprets (with parameters) T_1 in T_2 iff

$$\forall \varphi \in L^0_\tau \; (\varphi \in T_1 \iff (\forall \bar{x}\; \varphi_p(\bar{x}) \longrightarrow Fc(\varphi)) \in T_2$$

(b) T_1 is interpretable (with parameters) in T_2 iff there exists a code c from σ to τ such that c interprets T_1 in T_2.

(c) T_1 and T_2 are mutually interpretable (with parameters) iff T_1 is interpretable in T_2 and T_2 is interpretable in T_1.

It is easy to prove the following useful fact:
a code c interprets T_1 in T_2 iff $T_1 = Th(\bigcup\{\Gamma c\,\mathfrak{A} : \mathfrak{A} \models T_2\})$.
The proof of this lemma, basic properties and examples of introduced notions can be found in [4] or [5].

It is obvious, that the relation on mutual interpretability (df.4 c is the equivalence relation on the class of all theories. Classes of equivalence of this relation are called the domains.
Let us define:

$[T_1] \leq [T_2]$ iff T_1 is interpretable in T_2; where $[T]$ denotes the class of equivalence containing the theory T.
The relation $\leq$ partially orders the class of all domains and, moreover, the following conditions are satisfied:

1. There exists the minimal domain. This is the class of equivalence containing the theory, which has exactly one, one-element model.
2. $\leq$ is a linear-order on a set of all trivial domains (domain is trivial iff one of its elements has only one-element models)
3. Let D be the domain including the theory of two-elements set. Then for any nontrivial domain D', $D \leq D'$.
4. There exists an antichain of domain, which is of power

continuum.

5. Let D be an arbitrary domain. Then the set $\{D' : D' \leq D\}$ is at most countable.
6. There is no maximal domain.
7. Each finite set of domains has supremum.
8. Each countable set of fomains has an upper bound.
9. There exists countable set of domains, which has no supremum.
10. There exists a finite set of domains, which has no infinimum.

The proofs of all these facts will be published in the different paper.

REFERENCES

[1] Mostowski A., Robinson R., Tarski A., "Undecidable Theories"

[2] Mycielski J., "A lattice of interpretability types of theories" (J. Symb. Log. 42)

[3] Lindström P., "A lattice of degrees of interpretability" in Some results on interpretability"

[4] Szczerba L., "Interpretations with parameters" (Zeitschr. f. math. Logic und Grundlagen d. Math. 1980)

[5] Szczerba L., "Interpretacje z parametrami"

PEWNE WYNIKI DOTYCZĄCE INTERPRETOWALNOŚCI Z PARAMETRAMI

Streszczenie

Praca zwiera streszczenie wyników dociekań autorki nawiązujących do prac L. Szczerby z zakresu interpretowalności. Poza wprowadzeniem w tematykę artykuł niniejszy prezentuje pewną liczbę definicji potrzebnych do wprowadzenia pojęcia dziedziny. Następnie podano tu charakterystyczne własności dziedziny.