ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY

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ON IDEALS OF THE SETS ON THE REAL LINE WHICH CONTAIN THE IDEAL OF THE SETS OF POWER $< 2^{\omega_0}$

Let X denotes a set. If $A \subseteq X^2$ then $A_x = \{y \in X;\ (x,y) \in A\}$ for each x of X and let $A^y = \{x \in X:\ (x,y) \in A\}$ for each y of X. W. Sierpiński showed in [2] that the following statements are equivalent :

(i) the Continuum Hypothesis (CH),

(ii) there exists a partition $A, B \subseteq R^2$ (by R is always denoted the real line) such that $\forall x \in R (|A_x| \le \omega_0 \& |B^x| \le \omega_0)$

In this paper we generalize Sierpiński's result. It is shown in Theorem 0 that if $|X| = \aleph$ is a regular cardinal number and $I \subseteq P(X)$ is a proper ideal then the following statements are equivalent :

(i) $\forall A \subseteq X \quad (|A| < \aleph \implies A \in I)$,

(ii) there exists a partition $A, B \subseteq X^2$ such that

$$\forall x \in X \quad (|B_x| < \aleph \quad \& \quad A^x \in I).$$

We use standart set theoretical notation and therminology. A cardinal number $\aleph$ is identified with the first ordinal number of a power $\aleph$.
If X is a topological space then the sign $P(X)$ denotes the family of all subsets of X and the sign $\mathcal{B}(X)$ denotes the family of the Borel sets of X.
An ideal $I \subseteq P(X)$ is called a σ-ideal iff the union of countable many members of I always belongs to I.

An ideal $I \subseteq P(X)$ is called proper iff $X \notin I$.
A proper ideal $I \subseteq P(X)$ is called $\aleph$-semicomplete iff $\forall \alpha < \aleph \ (\{A_\xi : \xi < \alpha\} \subseteq I \implies \bigcup\{A_\xi : \xi < \alpha\} \neq X)$.

If $I \subseteq P(X)$ is an ideal then $\mathcal{B}^I(X)$ denotes the family

$\{A \subseteq X : \exists B \in \mathcal{B}(X) \quad A \triangle B \in I\}$, $(A \triangle B = (A-B) \cup (B-A))$.

Clearly $\mathcal{B}^I(X)$ is a field and $\mathcal{B}(X) \cup I \subseteq \mathcal{B}^I(X)$.

An ideal $\mathcal{J} \subseteq P(X^2)$ is a <u>second power</u> of an ideal $I \subseteq P(X)$ iff

(i) $\forall A \subseteq X^2 \ \{A \in \mathcal{J} \Longrightarrow \exists B_1, B_2 \in I \quad [\forall x \in X - B_1 (A_x \in I)$ & $\forall x \in X \setminus B_2 (A^x \in I)]\}$,

(ii) $\forall A \subseteq X^2 \ [A \in \mathcal{B}^{\mathcal{J}}(X^2) - \mathcal{J} \Longrightarrow \exists x,y \in X \ (A_x \in \mathcal{B}^I(X) \setminus I$ & $A^y \in \mathcal{B}^I(X) \setminus I)]$.

For every set X such as $|X| > \omega_0$ <u>$K(X)$</u> denotes the ideal of the sets of power $< |X|$.

Theorem 0.

Suppose, that $|X| = \aleph$, where $\aleph$ is a regular cardinal number. If $I \subseteq P(X)$ is a proper ideal then following statements are equivalent :

(i) $K(X) \subseteq I$,

(ii) there exists a partition $A, B \subseteq X^2$ such that

$$\forall x \in X \ (B_x \in K(X) \ \& \ A^x \in I)$$

Proof. (i) $\Longrightarrow$ (ii). Let $(a_\alpha)_{\alpha < \aleph}$ be an enumeration of X and $A_\alpha = \{a_\beta : \beta \le \alpha\}$ for $\alpha < \aleph$.

A, B are defined as follows :

$$A = \bigcup_{\alpha <} A_\alpha \times \{a_\alpha\}, \quad B = X^2 \setminus A.$$

Clearly, if $x = a_\alpha$ then $A^x = A_\alpha \in I$ and $B_x \subseteq A_\alpha$.

Hence $|B_x| < \aleph$.

(ii) $\Longrightarrow$ (i). If $C \subseteq X$ and $C \in K(X)$ then $C \subseteq A^{y_0}$ for some $y_0 \in X$. Really, suppose as there exists $C \in K(X)$ such that

$$\forall y \in X \quad C \times \{y\} \not\subseteq A.$$

Then $\forall y \in X \ \exists x \in C \ \langle x, y \rangle \in B$.

Since $C \in K(X)$ and $\aleph$ is a regular cardinal number then there exists $x_0 \in C$ such that

$$|\{y \in X : \langle x_0, y \rangle \in B\}| = \aleph ,$$

a contradiction with (ii).

Let $C \in K(X)$, then there exists $y_0 \in X$ such that $C \subseteq A^{y_0}$.

Since $A^{y_0} \in I$ then $C \in I$.

Remark 0.

Consider the implication (ii) $\Longrightarrow$ (i).

The assumption that $\aleph$ is regular is essential.

P r o o f. Suppose that $|X| = 2^{\omega_0} = \omega_{\omega_1}$. Let $(a_\alpha)_{\alpha < \omega_{\omega_1}}$ be an enumeration of X. Let for $\xi < \omega_1$

$$C_\xi = \{a_\alpha : \omega_\xi \leq \alpha < \omega_{\xi+1}\}.$$

Notice that $X = \bigcup_{\xi < \omega_1} C_\xi$. Let I be the σ-ideal generated by family $\{C_\xi : \xi < \omega_1\}$. Notice that I is proper.

$$I = \{A \subseteq X : A \subseteq \bigcup_{t \in T} C_t, \quad T \subseteq \omega_1, \ |T| \leq \omega_0\}.$$

Define the sets $A, B \subseteq X^2$:

$$B = \bigcup_{\xi < \omega_1} \bigcup_{\eta < \xi} C_\xi \times C_\eta, \quad A = X^2 - B.$$

If $x \in X$ then there exists $\xi < \omega_1$ such that $x \in C_\xi$.

Consequently $B_x = \bigcup_{\eta < \xi} C_\eta$, $|\xi| \leq \omega_0$, $|C_\eta| < \omega_{\omega_1}$ and cofinality of ω_{ω_1} is ω_1, so $|B_x| < \omega_{\omega_1}$.

Similary, $A^x \subseteq \bigcup_{\eta < \xi+1} C_\eta$ and $|\xi+1| = \omega_0$. Hence $A^x \in I$.

Let D be a selector for $\{C_\xi : \xi < \omega_1\}$. Then $|D| = \omega_1 < \omega_{\omega_1}$ and $D \notin I$ and therefore $K(X) \not\subseteq I$.

Theorem 1

Accepting the assumptions of the theorem 0 we suppose furthermore that I is a $\aleph$-semicomplete ideal. Then the following statements are equivalent :

(i) $K(X) \subseteq I$.

(iii) there exists a partition $A, B \subseteq X^2$ such that

$$\forall x \in X \quad (A^x \cup B_x \in I).$$

Proof. Clearly (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii)

Now, suppose that the statement (iii) holds.

Assume, as there exists $C \in K(X)$ such that $C \notin I$.

Let $C = \{c_\xi : \xi < \alpha\}$ for some $\alpha < \aleph$.

Since

$\forall x \in X \quad C \nsubseteq B_x$ then

$\forall x \in X \quad \exists y \in C \quad \langle x, y \rangle \in A$

Let $C_\xi = \{x \in X : \langle x, c_\xi \rangle \in A\}$.

Since $C \subseteq A^{c_\xi}$ then $C_\xi \in I$ for every $\xi < \alpha < \kappa$

and

$$X = \bigcup \{C_\xi : \xi < \alpha\} .$$

But I is semicomplete - a contradiction.

Remark 1. ($\neg$ CH)

Consider the implication (iii) $\Longrightarrow$ (i)

The assumption that I is κ-semicomplete is essential.

Proof. Suppose that $|X| = 2^{\omega_0} > \omega_1$. Let $\{C_\xi : \xi < \omega_1\}$ be a partition of X. Let I be the σ-ideal generated by this family. Notice that I is proper.

Define the sets $A, B \subseteq X^2$:

$$B = \bigcup_{\xi < \omega_1} \bigcup_{\eta < \xi} C_\xi \times C_\eta , \qquad A = X^2 - B .$$

If $x \in X$ then there exists $\xi < \omega_1$ such that $x \in C_\xi$. Then $B_x = \bigcup_{\eta < \xi} C_\eta$, $A^x = \bigcup_{\eta \leq \xi} C_\eta$, $|\xi| \leq \omega_0$ and $C_\eta \in I$. Hence $B_x \in I$ and $A^x \in I$. Let D be a selector for $\{C_\xi : \xi < \omega_1\}$. Clearly $D \in K(X) - I$.

Corollary 0. (Sierpiński)

If I is the ideal of countable subsets of R then the following statements are equivalent:

(i) CH ,

(ii) there exists a partition $A, B \subseteq R^2$ such that
$$\forall x \in R \; (|A^x| \leq \omega_0 \;\&\; |B_x| \leq \omega_0).$$

Corollary 1.

If I is the ideal of the Lebesgue measure 0 sets on the real line then the following statements are equivalent:

(i) $\forall A \subseteq R \; (|A| < 2^{\omega_0} \Longrightarrow A \in I)$,

(ii) there exists a partition $A, B \subseteq R^2$ such that
$$\forall x \in R \; (A^x \in I \;\&\; |B_x| < 2^{\omega_0}).$$

Corollary 2.

If I is the ideal of the first category sets on the real line then the following statements are equivalent:

(i) $\forall A \subseteq R(|A| < 2^{\omega_0} \Longleftrightarrow A \in I)$,

(ii) there exists a partition $A, B \subseteq R^2$ such that

$$\forall x \in R\ (A^x \in I \;\&\; |B_x| < 2^{\omega_0}).$$

Remark 2.

Consider the implication (ii) $\Longrightarrow$ (i).
The assumption that $\varkappa$ is regular is essential.

P r o o f. It is consistent with ZFC that $2^{\omega_0} = \omega_{\omega_1}$, the real line is the union of ω_1 many sets of the first category and $K(R) \notin I$ (comp. [3]).
Then the condition (ii) holds.
In fact, let $R = \bigcup\{C_\xi : \xi < \omega_1\}$ and $C_\xi \in I$ for every $\xi < \omega_1$. Suppose that the sets C_ξ are pairwise disjoint (if they are not then we define

$$C'_\xi = C_\xi - \bigcup_{\eta<\xi} C_\eta \text{ for } \xi < \omega_1) \text{ and } |C_\xi| \leq \omega_{\xi+1} \text{ for every } \xi < \omega_1.$$

Indeed, let $(a_\alpha)_{\alpha<\omega_{\omega_1}}$ be an enumeration of R.
Define

$$C'_\xi = C_\xi \cap \{a_\alpha : \alpha < \omega_{\xi+1}\} \cup \bigcup_{\eta<\xi} C_\eta \cap \{a_\alpha : \omega_\xi \leq \alpha < \omega_{\xi+1}\}.$$

Notice that $C'_\xi \in I$, $|C'_\xi| \leq \omega_{\xi+1}$ and $\bigcup_{\xi<\omega_1} C'_\xi = \bigcup_{\xi<\omega_1} C_\xi = R$.
Define

$$B = \bigcup_{\xi<\omega_1} \bigcup_{\eta<\xi} C_\xi \times C_\eta \quad \text{and} \quad A = R^2 - B.$$

Let $x \in R$ then $x \in C_\xi$ for some $\xi < \omega_1$. Therefore $B_x = \bigcup_{\eta<\xi} C_\eta$, $|B_x| < \omega_{\xi+1}$ and $A^x = \bigcup_{\eta<\xi+1} C_\eta \in I$.
Similarly, if I is the ideal of the Lebesgue measure 0 sets. If it is consistent with ZFC that $2^{\omega_0} = \omega_{\omega_1}$, the real line is the union of ω_1 many sets of I and $K(R) \notin I$ (which the author belives in), then the condition (ii) holds for I.

Corollary 3.

If I is the ideal of the Lebesgue measure 0 sets of R (of the first category sets of R) and I is 2^{ω_c}-semicomplete then the following statements are equivalent:

(i) $\forall A \subseteq R \quad (|A| < 2^{\omega_0} \Longrightarrow A \in I)$,

(ii) there exists a partition $A, B \subseteq R^2$ such that

$$\forall x \in R \; (A^x \in I \;\&\; B_x \in I).$$

Remark 3.

Let I be the ideal of the Lebesgue measure 0 sets of R (of the first category sets of R). Consider the implication (ii)$\Longrightarrow$(i).

The assumption that I is 2^{ω_0}-semicomplete is essential.

P r o o f. It is consistent with ZFC $2^{\omega_0} = \omega_2$, I is not 2^{ω_0}-semicomplete and $K(R) \notin I$ (comp. [3]). Then (ii) holds.

Indeed, let $\{C_\xi : \xi < \omega_1\}$ be a partition of real line such that $C_\xi \in I$ for every ξ.

Define $B = \bigcup_{\xi < \omega_1} \bigcup_{\eta < \xi} C_\xi \times C_\eta$, $A = R^2 - B$.

It is clear that the partition A, B satisfies the condition (ii).

Remark 4. (ZFC)

Suppose that I is a proper ideal such that $K(X) \cap \mathcal{B}^I(X) \subseteq I$, J is a second power of I and A, B satisfy the condition (ii). Then $A, B \notin \mathcal{B}^J(X^2)$.

P r o o f. Let A, B be a partition of X^2 such that

$$\forall x \in X \; (B_x \in K(X) \;\&\; A^x \in I).$$

Suppose that $A \in \mathcal{B}^J(X^2)$. Since $A^x \in I$ for every $x \in X$, then $A \in J$. Since $A \in \mathcal{B}^J(X^2)$, then $B \in \mathcal{B}^J(X^2)$. Consequently $B \in J$. In fact, if $B \notin J$ then there exists x such that $B_x \in \mathcal{B}^I(X) - I$, but $B_x \in K(X)$ - a contradiction with $K(X) \cap \mathcal{B}^I(X) \subseteq I$.

Thus $A, B \in J$, $X^2 = A \cup B \in J$ and $X \in I$ - a contradiction, because I is proper.

Remark 5. (ZFC)

If I is a proper ideal, J is a second power of I and A, B satisfy the condition (iii) then $A, B \notin \mathcal{B}^J(X^2)$.

It is clear.

Corollary 5. (ZFC)
If I is the ideal of the Lebesgue measure 0 sets of R, J is the ideal of the Lebesgue measure 0 sets of R^2, and A, B satisfy the condition (ii) or (iii) then $A,B \notin \mathcal{B}^J(R^2)$.

Corollary 6. (ZFC)
If I is the ideal of the first category sets of R, J is the ideal of the first category sets of R^2 and A,B satisfy the condition (ii) or (iii) then $A,B \notin \mathcal{B}^J(R^2)$.

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O IDEAŁACH NA PROSTEJ ZAWIERĄJACYCH IDEAŁ PODZBIORÓW MOCY$<2^{\omega}$.

Streszczenie

W. Sierpiński udowodnił równoważność następujących warunków:

(i) Hipoteza Continuum,

(ii) istnieje podział płaszczyzny na dwa zbiory, z których jeden ma wszystkie cięcia poziome przeliczalne, zaś drugi wszystkie cięcia pionowe przeliczalne.

W pracy niniejszej rozważa się możliwość zastąpienia w twierdzeniu Sierpińskiego ideału zbiorów przeliczalnych przez inne ideały (zawierające ideał zbiorów mocy mniejszej niż continuum).