ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY

Problemy Matematyczne 1983/84 z. 5/6

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AN ESTIMATION OF THE SOLUTION OF AN OPERATOR INTEGRAL EQUATION OF VOLTERRA'S TYPE FOR VECTOR- VALUED FUNCTION WITH VALUES IN AN ORLICZ-SOBOLEV SPACE WITH ARBITRARY MEASURE

1. In [2] we dealt with a Volterra's integral equation

$$(1)\qquad u(x) = \int_a^x T(x,t)u(t)dt + b(x)$$

where $x, t, a \in R^n$, b and u are vector-valued functions of the variable t $(a \le t \le x)$ with values in a Banach space Y and $T(x,t)$ is a linear bounded operator of Y into itself for $a \le t \le x$, strongly measurable in both variables. The order relation $c \le d$ for $c = (c_1,\dots,c_n) \in R^n$, $d = (d_1,\dots,d_n) \in R^n$ means here that $c_i \le d_i$ for $i=1,2,\dots,n$. Denoting by $\mathcal{Y}$ the space of all such operators and supposing that $\|b(x)\|_Y \le B(x)$ where $B(x)$ is measurable and bounded for $x \ge a$, there was proved the following theorem:

Let us suppose that $A(x,t)$ is a real-valued function, defined for $a \le t \le x$, nondecreasing with respect to x for every t and such that $A(t,t)$ is measurable and bounded for $t \ge a$, $\alpha \int_a^x A(t,t)dt < 1$ for $x \ge a$. Moreover, let us suppose that

$$(2)\qquad \|T(x,t)\|_{\mathcal{Y}} = \alpha A(x,t) \qquad \text{for} \quad a \le t \le x,$$

where α is independent of x and t.

Then the integral equation (1) has a unique solution in the space of Y-valued, bounded and strongly measurable functions in $x \ge a$. Moreover, we have an estimation

$$\|u(x)\|_Y \le \beta(x) \operatorname{esp} \left(\int_a^x \alpha A(t,t)dt \right) \quad \text{for} \quad x \ge a \quad ,$$

where $\beta(x) = \sup_{a \le t \le x} B(t)$, and where $\|b(x)\|_Y \le B(x)$.

2. In [3] the case $Y = W_\varphi^k(E)$ Orlicz-Sobolev space were considered, where E denoted a bounded open subset of R^n and where φ denoted a convex, nondecreasing real function, φ^* the function complementary to φ in the sense of Young and $W_\varphi^k(E)$ the space of measurable functions with generalized derivatives $D^\beta u \in L^\varphi(E)$ for $|\beta| \le k$, and $L^\varphi(E)$ is an Orlicz space. In [3] there was given an estimation for the norm of an integral operator T of the form $v=Tu$, where $v_\sigma = \int_E T_{\sigma,\tau} u_\tau \, d\tau$, in case of function u from the Orlicz-Sobolev space $W_\varphi^k(E)$ over an open subset E of R^n of finite measure. Writing $T=T(x,t)$ whith fixed $x,t \in E$, this estimation was of the form $\|T\|_{\hat{\mathcal{Y}}} \le \alpha A^\wedge$, where $\mathcal{Y}$ is the space of linear, continuous operators in $W_\varphi^k(E)$,

$\|T\|_{\hat{\mathcal{Y}}} = \sup_{\|u\|_{\hat{\varsigma}} \le 1} \|Tu\|_{\hat{\varsigma}}$ with

$$\|u\|_{\hat{\varsigma}} = \sum_{|\beta| \le K} \|D^\beta u\|_{\tilde{\varsigma}} \quad ; \quad \tilde{\varsigma}(u) = \int_E \varphi(|u_\tau|)\, d\tau ,$$

and $T_{\sigma,\tau}$ satisfy the following conditions:

1^o $D_\sigma^\beta(T_{\sigma,\tau})$ exist in the usual sense for $\sigma, \tau \in E$, $\beta \le K$ and are continuous functions of the variable $\sigma \in E$ uniformly with respect to $\tau \in E$

2^o $D_\sigma^\beta(T_{\sigma,\tau}) \in L\varphi^*(E)$ as a function of variable $\tau \in E$ for every $\sigma \in E, |\beta| \le K$

3^o $|D_\sigma^\beta(T_{\sigma,\tau})| \le A_\beta$ for $\sigma, \tau \in E$, $|\beta| \le K$.

4^o $A^\wedge = \sum_{|\beta| \le K} A_\beta$

3. Here we are going to extend this result to the case of an open set $E \subset R^n$ of arbitrary measure possibly also infinite. The assumptions 1^o and 3^o will be now replaced by the following ones:

$1'$ 1^o holds in every set of finite measure $E_1 \subset E$

$3'$ $|D_\sigma^\beta(T_{\sigma,\tau})| \le e_\beta(\sigma)\varepsilon_\beta(\tau) \cdot A_\beta$ for $|\beta| \le K$ where $0 < e_\beta(\sigma) \le 1$ in E and $0 < \varepsilon_\beta(\tau) \in L\varphi^*$ for $|\beta| \le K$, being complementary to φ in the sense of Young.

Let us write $M = \operatorname{Max} \|\mathcal{E}_\beta\|_{\tilde{\varsigma}^*}$, $A = \max_{|\beta|\le k} A_\beta$, where $\tilde{\varsigma}^*(u) = \int_E \varphi^*(|u_\tau|)d\tau$.

First, let us observe that the following lemma holds.

__Lemma.__ Let φ satisfy the condition

(Δ_2) $\varphi(2u) \le K\varphi(u)$ for $u \ge 0$.

Assuming $T_{\sigma,\tau}$ satisfies the assumptions $1'$, 2^o and $3'$, there holds

$$(*)\ D_\sigma^\beta\left(\int_E T_{\sigma,\tau} u_\tau\, d\tau\right) = \int_E D_\sigma^\beta (T_{\sigma,\tau}) u_\tau\, d\tau \quad \text{for every}$$

$u \in L^\varphi(E)$.

__Proof.__ Existence of the integral at the ringht-hand side of $(*)$ follows from the inequality

$$\int_E |D_\sigma^\beta (T_{\sigma,\tau})u_\tau|\,d\tau \le \|D_\sigma^\beta(T_{\sigma,\cdot})\|_{\tilde{\varsigma}^*}\ \|u\|_{\tilde{\varsigma}}^{o} ,$$

where $\|\cdot\|_{\tilde{\varsigma}}^{o}$ is the Orlicz norm in $L^\varphi(E)$ and $\tilde{\varsigma}(u) = \int_E \varphi(|u_\tau|)d\tau$ is the modular defining the Orlicz space $L^\varphi(E)$, $\tilde{\varsigma}^*$ is the modular defined as above, where φ^* is the complementary function to φ in the sense of Young.

Obviously, $(*)$ is true for $\beta = (0,0,\dots,0)$. Now let us suppose that $(*)$ holds for $|\beta| \le 1$ if $0 \le 1 < k$. Let $|\beta| \le 1$, $\beta = (\beta_1,\dots,\beta_i,\dots,\beta_\mu)$, $\beta' = (\beta_1,\dots,\beta_i+1,\dots,\beta_u)$, $\bar{h} = (0,\dots,h,\dots,o)$ with $h > 0$ at the i-th place. Let $\mathcal{E} > 0$ be given. There exists a $\delta > 0$ such that $|D^{\beta'}(T_{\sigma+\bar{h},\tau}) - D^{\beta'}(T_{\sigma,\cdot})| < \mathcal{E}$ for $|h| < \delta$ and all σ such that σ, $\sigma + \bar{h} \in E$.

Let us write

$$\Delta_h = \frac{1}{h}\int_E \left[D_\sigma^\beta (T_{\sigma+\bar{h},\tau}) - D_\sigma^\beta(T_{\sigma,\tau}) - D_\sigma^{\beta'}(T_{\sigma,\tau})h\right] u_\tau\, d\tau$$

where β, β', $\bar{h}$ are defined as above, $0 < |\beta| \le 1 < k$.

Applying the mean-value theorem, we get

$$|\Delta_h| = \int_E |D_\sigma^{\beta'}(T_{\sigma+\vartheta\bar{h},\tau}) - D_\sigma^{\beta'}(T_{\sigma,\tau})|\,|u_\tau|\,d\tau \quad \text{with } 0 < \vartheta < 1.$$

We split now the last integral over E onto integrals I_1 and I_2 over disjoint sets E_1 and E_2 such that $E = E_1 \cup E_2$ and E_1 is of finite measure obtain $|\Delta_h| \le I_1 + I_2$. It is easily seen that

$$I_2 \le \int_{E_2} |D_\sigma^{\beta'}(T_{\sigma+\vartheta\bar{h},\tau})|\,|u_\tau|\,d\tau + \int_E |D_\sigma^{\beta'}(T_{\sigma,\tau})|\,|u_\tau|\,d\tau \le$$

$$\le (\|D_\sigma^{\beta'}(T_{\sigma+\vartheta\bar{h},\cdot})\|_{\tilde{\varsigma}^*} + \|D_\sigma^{\beta'}(T_{\sigma,\cdot})\|_{\tilde{\varsigma}^*})\,\|u\chi_{E_2}\|_{\varsigma}^{\circ} ,$$

where χ_{E_2} is the characteristic function of the set E_2. But by assumption $3'$, where we have $0<\varepsilon_\beta(\sigma)\le 1$ in E, $\|D_\sigma(T_{\sigma,\cdot})\|_{\tilde{\varsigma}^*} \le \|\varepsilon_{\beta'}\|_{\tilde{\varsigma}^*} A_{\beta'}$; $\|D_\sigma^{\beta'}(T_{\sigma+\vartheta\bar{h},\cdot})\|_{\tilde{\varsigma}^*} \le \|\varepsilon_{\beta'}\|_{\tilde{\varsigma}^*} A_{\beta'}$.

Hence

$$I_2 \le 2A_{\beta'}\|\varepsilon_{\beta'}\|_{\tilde{\varsigma}^*}\|u\chi_{E_2}\|_{\varsigma}^{\circ} \le 2AM\,\|u\chi_{E_2}\|_{\varsigma}^{\circ} \le 4AM\,\|u\chi_{E_2}\|_{\varsigma} .$$

Let $\varepsilon > 0$ be given and let us take $\eta = \frac{\varepsilon}{8AM}$. Since $u \in L^{\varphi}(E)$ and φ satisfies (Δ_2) for $u \ge 0$, so

$$\int_E \varphi\left(\frac{1}{\eta}|u_\tau|\right)d\tau < +\infty .$$

But we have $E=E_1 \cup E_2$, where the set $E_1 \subset E$ is of finite measure and therefore writing $E_2 = E \setminus E_1$ we see easily that

$$\int_{E_2} \varphi\left(\frac{1}{\eta}|u_\tau|\right)d\tau < 1$$

and consequently $\|u\chi_{E_2}\|_{\varsigma} < \eta$. Thus $I_2 < \frac{1}{2}\varepsilon$. Now keeping E_1 fixed from the assumption $1'$ we conclude that there exists a $\delta > 0$ such that

$$|D^{\beta'}(T_{\sigma+\bar{h},\tau}) - D^{\beta'}(T_{\sigma,\tau})| < \frac{\varepsilon}{2\|\chi_{E_1}\|_{\tilde{\varsigma}^*}\|u\|_{\varsigma}^{\circ}}$$

for $\tau \in E_1$ and $|h| < \delta$, where σ, $\sigma+\bar{h} \in E_1$. Hence

$$I_1 \le \|D_\sigma^{\beta'}(T_{\sigma+\vartheta\bar{h},\cdot}) - D_\sigma^{\beta'}(T_{\sigma,\cdot})\|\,\|\chi_{E_1}\|_{\tilde{\varsigma}^*}\|u\|_{\varsigma}^{\circ} \le$$

$$\le \frac{\varepsilon}{2\|\chi_{E_1}\|_{\tilde{\varsigma}^*}\|u\|_{\varsigma}^{\circ}}\,\|1\|_{\tilde{\varsigma}^*}\|u\|_{\varsigma}^{\circ} = \frac{1}{2}\varepsilon \quad \text{for } |h| < \delta .$$

Consequently, $|\Delta_h| < \varepsilon$ for $|h| < \delta$. This proves the lemma.

<u>Theorem.</u> Let us suppose that $T_{\sigma,\tau}$ satisfies the assumptions $1'$, 2^0, $3'$ and 4^0 with $0 < \varepsilon_\beta(\tau) < 1$. Additionally, let the functions $e_\beta(\sigma)$ and $\varepsilon_\beta(\tau)$ from $3'$ be integrable over E and $0 < \varepsilon_\beta(\tau) \le 1$ for $\tau \in E$. Then $\|T\|_{\gamma}^{\wedge} \le \alpha A^{\wedge}$ with $\alpha = C'$, where

$$C' = \max_{|\beta| \le \varkappa} \left\{ \max \left(\int_E e_\beta(\sigma) d\sigma , \int_E \varepsilon_\beta(\tau)\, d\tau \right) \right\} .$$

<u>**Proof.**</u> Let $\vartheta(A) = \int_A \varepsilon(\tau) d\tau$, $\lambda(A) = \int_A e(\sigma) d\sigma$ for measurable $A \subset E$. Then we have with $0 < \varepsilon_\beta(\tau) < 1$

$$\tilde{\varsigma}(D^\beta v) = \int_E \varphi(|D^\beta_\sigma v_\sigma|) d\sigma = \int_E \varphi\left[D^\beta_\sigma \left(\int_E T_{\sigma,\tau} u_\tau\, d\tau\right)\right] d\sigma =$$

$$= \int_E \varphi\left[\int_E D^\beta_\sigma (T_{\sigma,\tau}) u_\tau\, d\tau\right] d\sigma \le \int_E \varphi\left[\int_E e_\beta(\sigma)\varepsilon_\beta(\tau) A_\beta\, u_\tau\right] d\sigma \le$$

$$\le \int_E e_\beta(\sigma) d\sigma \cdot \varphi\left(A_\beta \int_E u_\tau \varepsilon_\beta(\tau) d\tau\right) = \lambda(E)\, \varphi\left(A_\beta \int_E u_\tau \vartheta(d\tau)\right) =$$

$$= \lambda(E)\, \varphi\left(\frac{1}{\vartheta(E)} \int_E A_\beta\, \vartheta(E)\, u_\tau \cdot \vartheta(d\tau)\right) \le \lambda(E)\, \frac{1}{\vartheta(E)} \int_E \varphi(A_\beta \vartheta(E) u_\tau) \vartheta(d\tau).$$

Then we obtain

$$\tilde{\varsigma}(D^\beta v) = \frac{\lambda(E)}{\vartheta(E)} \int_E \varphi(A_\beta \vartheta(E) u_\tau)\, \vartheta(d\tau).$$

Let us consider two cases:

1) $0 < \lambda(E) \le \vartheta(E)$

$$\tilde{\varsigma}(D^\beta v) \le \int_E \varphi(A_\beta \vartheta(E) u_\tau)\, \vartheta(d\tau) = \int_E \varepsilon(\tau) \varphi(A_\beta \vartheta(E) u_\tau) d\tau \le$$
$$\le \int_E \varphi(A_\beta \vartheta(E) u_\tau) d\tau$$

2) $\lambda(E) > \vartheta(E)$ that is $\Lambda = \dfrac{\lambda(E)}{\vartheta(E)} > 1$, then $\varphi(\Lambda w) \ge \Lambda \varphi(w)$.

Therefore

$$\tilde{\varsigma}(D^\beta v) \le \Lambda \int_E \varphi(A_\beta \vartheta(E) u_\tau)\, \vartheta(d\tau) \le$$

$$\le \int_E \varphi\left(\frac{\lambda(E)}{\vartheta(E)} A_\beta \vartheta(E) u_\tau\right) \vartheta(d\tau) = \int_E \varphi(\lambda(E) A_\beta\, u_\tau)\, \vartheta(d\tau) =$$
$$= \int_E \varepsilon(\tau)\, \varphi(\lambda(E) A_\beta u_\tau) d\tau \le \int_E \varphi(\lambda(E) A_\beta u_\tau) d\tau$$

Hence

$$\tilde{\varsigma}(h^\beta \vartheta) \le \int_E \varphi(A_\beta \max(\lambda(E)\,;\, \vartheta(E))\, u_\tau) d\tau =$$
$$= \int_E \varphi(C' A_\beta u_\tau) d\tau = \tilde{\varsigma}(C' A_\beta u)$$

Consequently,

$$\|D^{\beta} v\|_{\varsigma} \leq \|C' A_{\beta} u\|_{\tilde{\varsigma}} = C' A_{\beta} \|u\|_{\tilde{\varsigma}}$$

$$\|T\|_{\hat{\gamma}} = \sup_{\|u\|_{\hat{\varsigma}} \leq 1} \sum_{\beta} \|D^{\beta}_{\varsigma} v\|_{\tilde{\varsigma}} \leq \sup_{\|u\|_{\hat{\varsigma}} \leq 1} \sum_{\beta} C' A_{\beta} \|u\|_{\tilde{\varsigma}} = C' \sum_{\beta} A_{\beta} \|u\|_{\tilde{\varsigma}} = C' \hat{A}.$$

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OSZACOWANIE ROZWIĄZANIA OPERATOROWEGO RÓWNANIA CAŁKOWEGO VOLTERRY DLA FUNKCJI WEKTOROWYCH O WARTOŚCIACH W PRZESTRZENIACH ORLICZA-SOBOLEWA Z DOWOLNĄ MIARĄ

Streszczenie

W artykule oszacowania rozwiązania równania

$$u(x) = \int_a^x T(x,t)u(t)dt + b(x) \qquad (1)$$

są przeniesione na przypadek funkcji wektorowych o wartościach w przestrzeni Orlicza-Sobolewa $W_{\varphi}^{K}(E)$, gdzie zbiór otwarty E może być dowolnej (również nieskończonej) miary .