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POINTWISE APPROXIMATION OF FUNCTIONS BY INTEGRALS OF THE ABEL TYPE

1. Preliminaries. Let L be the class of all 2π-periodic complex-valued functions Lebesgue-integrable in the interval $\langle -\pi, \pi \rangle$. For any function $f \in L$, with Fourier coefficients a_0, a_k, b_k, we define the generalized Abel means

$$(1)\quad H_r[f](x)= \frac{a_0}{2} + \sum_{k=1}^{\infty}(1+(1-r)kh(r))r^k(a_k\cos kx + b_k\sin kx),$$

$$(2)\quad \tilde{H}_r[f](x)= \sum_{k=1}^{\infty} (1+(1-r)kh(r))r^k(a_k\sin kx - b_k\cos kx)$$

$(0<r<1)$, where h is a suitable real-valued, bounded function of r defined in $\langle r_0, 1\rangle$, $(0<r_0<1)$. The uniform approximation of continuous functions by the means (1) was examined in [2]. In the case $h(r)= \frac{1+r}{2}$ the operator H_r [resp. $\tilde{H}_r$] coincides with biharmonic [conjugate biharmonic] operator $B_r[\tilde{B}_r]$. Moreover, we introduce the Weierstrass means

$$(3)\quad W_r[f](x)= \frac{a_0}{2} + \sum_{k=1}^{\infty} r^{k^2} (a_k\cos kx + b_k \sin kx)\quad (0<r<1).$$

Given any $f \in L$, let us consider the points x for which

$$(4)\quad \int_0^t (f(x+u)+ f(x-u)- 2f(x))du = o(t) \quad \text{as} \quad t \to 0$$

or

$$(5)\quad \int_0^t (f(x+u)- f(x-u))du = o(t) \quad \text{as} \quad t \to 0 .$$

Write, for $\delta>0$,

$$w_x(\delta)=w_x(\delta;f)= \sup_{0<|t|\le\delta} \left| \frac{1}{2t} \int_0^t (f(x+u)+f(x-u)-2f(x))du \right| ,$$

and

$$\bar{w}_x(\delta)=\bar{w}_x(\delta;f)=\sup_{0<|t|\le\delta}\left|\frac{1}{2t}\int_0^t (f(x+u)-f(x-u))du\right| ,$$

respectively. Clearly, these quantities are non-decreasing functions of δ and by the well-known Lebesgue theorem

$$\lim_{\delta\to 0+} w_x(\delta;f)=0 \quad , \quad \lim_{\delta\to 0+} \bar{w}_x(\delta;f)=0$$

for almost every x.

In this paper we give estimates of the rate of convergence of functions (1), (3)[resp. (2)] at the points for which the condition (4)[resp. (5)] holds. Also, we shall show that our main results cannot be essentially improved. For this purpose we introduce the class $L(\Omega)$ of all functions $f\in L$ for which

$$\sup_{0<h\le\pi}\frac{w_x(h;f)}{\Omega(h)}\le 1 \quad \text{at a fixed } x,$$

where Ω is a non-negative and increasing real-valued function defined on $\langle 0,\pi\rangle$ with $\Omega(0)=0$.

For convenience, the suitable positive absolute constants, will be denoted by C_j $(j=1,2,3,\ldots)$.

2. Properties of H_r and $\tilde{H}_r$. Let us start with the following

THEOREM 1. If $f\in L$ and $r\in\langle\frac{2}{3},1)$, then

$$|H_r[f](x)-f(x)|\le\frac{1-r}{(1+r)^2}\,|1+r-2rh(r)|\,w_x(\pi;f)+$$

$$+\left(8+\frac{121}{4}\pi^5\right)(1-r)^3\int_{1-r}^{\pi}\frac{w_x(t;f)}{t^4}dt+\frac{11}{16}\pi^3(1-r)^2\int_{1-r}^{\pi}\frac{w_x(t;f)}{t^2}dt+$$

$$+\frac{3}{8}\pi^3(1-r)\,|1-h(r)|\int_{1-r}^{\pi}\frac{w_x(t;f)}{t^2}dt .$$

Proof. It is easily seen,

$$H_r[f](x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)J_r(t)dt ,$$

where

$$J_r(t) = \frac{1}{2} + \sum_{k=1}^{\infty} (1+(1-r)h(r)k)r^k \cos kt.$$

The kernel J_r can be rewritten in the form

$$J_r(t) = K_r(t) + (1-r)\left(h(r) - \frac{1+r}{2}\right)Z_r(t),$$

with

$$Z_r(t) = \sum_{k=1}^{\infty} kr^k \cos kt = r\,\frac{(1+r^2)\cos t - 2r}{(1-2r\cos t + r^2)^2}$$

and

$$K_r(t) = \frac{1}{2} + \sum_{k=1}^{\infty}\left(1 + \frac{k}{2}(1-r^2)\right)r^k \cos kt = \frac{(1-r^2)^2(1-r\cos t)}{2(1+r^2 - 2r\cos t)^2}.$$

Putting $\Phi_x(t) = \int_0^t \varphi_x(v)dv$, where $\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$, we observe that $|\Phi_x(t)| \le 2tw_x(t)$, $(t > 0)$.

Integrating by parts, we obtain

$$H_r[f](x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)J_r(t)dt = \frac{1}{\pi}\Phi_x(\pi)J_r(\pi) - \frac{1}{\pi}\int_0^{\pi}\Phi_x(t)K_r'(t)dt +$$

$$- \frac{1}{\pi}(1-r)\left(h(r) - \frac{1+r}{2}\right)\int_0^{\pi}\Phi_x(t)Z_r'(t)dt = A_1 + A_2 + A_3, \text{ say.}$$

Evidently,

$$(6)\quad |A_1| \le 2w_x(\pi)\left|\frac{1}{2}\,\frac{1-r^2}{(1+r)^2} - \frac{h(r)(1-r)r}{(1+r)^2}\right| = w_x(\pi)\frac{1-r}{(1+r)^2}|1+r-2rh(r)|.$$

By the estimates given in [3] for biharmonic operator,

$$(7)\quad |A_2| \le (8+\pi^5)(1-r)^3\int_{1-r}^{\pi}\frac{w_x(t)}{t^4}dt + \frac{\pi^3}{2}(1-r)^2\int_{1-r}^{\pi}\frac{w_x(t)}{t^2}dt.$$

Since

$$Z_r'(t) = -r\sin t\,\frac{(1-r)^2(r^2+4r+1) - 4r(1+r^2)\sin^2(t/2)}{(1-2r\cos t + r^2)^3},$$

we have

$$|A_3| \le \frac{1}{\pi}(1-r)\left|h(r) - \frac{1+r}{2}\right|\left(\int_0^{t_0} + \int_{t_0}^{\pi}\right)|\Phi_x(t)|$$

$$\times\left| r\sin t\,\frac{(1-r)^2(r^2+4r+1)-4r(1+r^2)\sin^2(t/2)}{(1-2r\cos t+r^2)^3}\right| dt\ ,$$

where $t_o=2\arcsin\left(\frac{1-r}{2}\sqrt{\frac{r^2+4r+1}{r(1+r^2)}}\right)$ is the point in the interval $(0,\pi)$ such that $Z_r'(t_o)=0$. Further,

$$|A_3|\le\frac{1}{\pi}(1-r)^3 6r\,\left|h(r)-\frac{1+r}{2}\right|\int_0^{t_o}|\Phi_x(t)|\,\frac{\sin t}{(1-r)^6}\,dt+$$

$$+\frac{1}{\pi}4r^2(1+r^2)(1-r)\left|h(r)-\frac{1+r}{2}\right|\int_{t_o}^{\pi}|\Phi_x(t)|\,\frac{\sin t\sin^2(t/2)}{4^3r^3\sin^6(t/2)}\,dt\le$$

$$\le\frac{12r}{\pi(1-r)^3}\left|h(r)-\frac{1+r}{2}\right|\int_0^{t_o}t^2w_x(t)dt\ +$$

$$+\frac{\pi^3}{8r}(1+r^2)(1-r)\left|h(r)-\frac{1+r}{2}\right|\int_{t_o}^{\pi}\frac{w_x(t)}{t^2}\,dt\ .$$

It is easy to see that $1-r<t_o\le\pi(1-r)$ when $\frac{2}{3}\le r<1$.

Consequently,

$$(8)\quad |A_3|\le 4\pi^2\left|h(r)-\frac{1+r}{2}\right|w_x(\pi(1-r))+$$

$$+\frac{3}{8}\pi^3(1-r)\left|h(r)-\frac{1+r}{2}\right|\int_{1-r}^{\pi}\frac{w_x(t)}{t^2}\,dt\ .$$

Combining (6), (7), (8) and using the estimate

$$(1-r)^3\int_{1-r}^{\pi}\frac{w_x(t)}{t^4}\,dt\ \ge\ \frac{8}{27\pi^3}\,w_x(\pi(1-r)),$$

we get the desired assertion.

REMARK 1. Consider now the class $L(\Omega)$. By Theorem 1,

$$\sup_{f\in L(\Omega)}|H_r[f](x)-f(x)|\le C_1(1-r)\,|1-h(r)|\,\Omega(\pi)+C_2(1-r)^3\int_{1-r}^{\pi}\frac{\Omega(t)}{t^4}\,dt+$$

$$+\ C_3(1-r)^2\int_{1-r}^{\pi}\frac{\Omega(t)}{t^2}\,dt\ +\ C_4(1-r)\,|1-h(r)|\int_{1-r}^{\pi}\frac{\Omega(t)}{t^2}\,dt\ .$$

The 2π-periodic function g defined by the formula $g(t)=\Omega(|t-x|)$ if $|t-x|\le\pi$ is of class $L(\Omega)$. Consequently, for every operator H_r with positive kerner J_r, we have

$$\sup_{f\in L(\Omega)} |H_r[f](x)-f(x)| \geq |H_r[g](x)| \geq$$

$$\geq \frac{2}{\pi}\int_{\pi(1-r)}^{\pi} \Omega(t)J_r(t)dt = \frac{2}{\pi}\int_{(1-r)}^{\pi} \Omega(t)U_r(t)dt +$$

$$+\frac{2}{\pi}(1-r)\left(h(r)-\frac{1+r}{2r}\right)\int_{\pi(1-r)}^{\pi} \Omega(t)Z_r(t)dt = I_1 + I_2 ,$$

where $U_r(t)=\frac{1}{2}+\sum_{k=1}^{\infty}\left(1+\frac{k}{2r}(1-r^2)\right)r^k\cos kt =$

$$= \frac{(1-r)^3(1+r)(1+\cos t)}{2(1-2r\cos t+r^2)^2} .$$

Arguing similarly as in [3] (Remark to Theorem 1) we easily get

$$I_1 \geq C_5(1-r)^3 \int_{\pi(1-r)}^{\pi/2} \frac{\Omega(t)}{t^4}\,dt .$$

Observing that $Z_r(t)\leq 0$ for $t\in\langle\pi(1-r),\pi\rangle$ and assuming

$$-\frac{1+r}{2r} < h(r) \leq \frac{1+r}{2r}$$

[cf. [2], Lemma 2], we obtain

$$I_2 \geq C_6(1-r)|1-h(r)| \int_{\pi(1-r)}^{\pi} \frac{\Omega(t)}{t^2}\,dt .$$

From the above considerations it follows that, for positive operators H_r, the estimate in Theorem 1 cannot be essentially improved.

COROLLARY 1. If $1-h(r)=O(1-r)$ as $r \to 1-$, then

$$|H_r[f](x)-f(x)| \leq C_7(1-r)^2 w_x(\pi;f) + C_8(1-r)^3\int_{1-r}^{\pi}\frac{w_x(t;f)}{t^4}\,dt +$$

$$+ C_9(1-r)^2\int_{1-r}^{\pi}\frac{w_x(t;f)}{t^2}\,dt$$

for all $r\in\langle\frac{2}{3},1)$.

COROLLARY 2. If $1-h(r)=O(1)$ as $r \to 1-$, then

$$|H_r[f](x)-f(x)|\leq C_{10}(1-r)w_x(\pi;f)+C_{11}(1-r)\int_{1-r}^{\pi}\frac{w_x(t;f)}{t^2}dt$$

for all $r\in\langle\frac{2}{3},1)$. In particular, when $h(r)\equiv 0$, we obtain the Hsiang Wen-Hang result ([4]).

THEOREM 2. Suppose that $f\in L$ and $r\in\langle\frac{1}{2},1)$. Then

$$\left|\tilde{H}_r[f](x)+\frac{1}{\pi}\int_{\pi(1-r)}^{\pi}\frac{f(x+t)-f(x-t)}{2\tan(t/2)}dt\right|\leq$$

$$\leq 345\pi^3(1-r)^2\int_{\pi(1-r)}^{\pi}\frac{\bar{w}_x(t;f)}{t^3}dt+40\pi^3(1-r)^2\left|h(r)-\frac{1+r}{2}\right|\int_{\pi(1-r)}^{\pi}\frac{\bar{w}_x(t;f)}{t^3}dt$$

Proof. Clearly,

$$\left|\tilde{H}_r[f](x)+\frac{1}{\pi}\int_{\pi(1-r)}^{\pi}\frac{f(x+t)-f(x-t)}{2\tan t(t/2)}dt\right|\leq$$

$$\leq\left|\tilde{H}_r[f](x)-\tilde{B}_r[f](x)\right|+\left|\tilde{B}_r[f](x)+\frac{1}{\pi}\int_{\pi(1-r)}^{\pi}\frac{f(x+t)-f(x-t)}{2\tan(t/2)}dt\right|=$$

$$=Y_1+Y_2\ .$$

By Theorem 3 of [3],

$$Y_2\leq C_{12}(1-r)^2\int_{\pi(1-r)}^{\pi}\frac{\bar{w}_x(t)}{t^3}dt\ ,$$

with $C_{12}\leq 345\pi^3$. We can easily observe that

$$Y_1=\frac{1}{\pi}\left|h(r)-\frac{1+r}{2}\right|(1-r)\left|\int_{-\pi}^{\pi}f(x+t)\sum_{k=1}^{\infty}kr^k\sin kt\,dt\right|=$$

$$=\frac{1}{\pi}\left|h(r)-\frac{1+r}{2}\right|(1-r)^2(1+r)r\left|\int_0^{\pi}(f(x+t)-f(x-t))R_r(t)dt\right|,$$

with $R_r(t)=\dfrac{\sin t}{(1-2r\cos t+r^2)^2}$.

Let

$$\Psi_x(t)=f(x+t)-f(x-t)\quad\text{and}\quad\Psi_x(t)=\int_0^t\Psi_x(v)dv\ .$$

Clearly,

$$|\Psi_x(t)| \le 2t\bar{w}_x(t;f).$$

The partial integration leads to

$$\int_0^{\pi}(f(x+t)-f(x-t))R_r(t)dt = -\int_0^{\pi}\Psi_x(t)R_r'(t)dt =$$

$$= -\Big(\int_0^{\pi(1-r)} + \int_{\pi(1-r)}^{\pi}\Big)\Psi_x(t)R_r'(t)dt = J_1 + J_2 .$$

Since

$$R_r'(t)= \frac{\cos t\,[(1-r)^2+4r\sin^2(t/2)]-4r\sin^2 t}{((1-r)^2 + 4r\sin^2(t/2))^3},$$

we have

$$|J_1| \le \int_0^{\pi(1-r)} \Psi_x(t) \frac{(1-r)^2}{((1-r)^2+ 4r\sin^2(t/2))^3}\, dt +$$

$$+ \int_0^{\pi(1-r)} \Psi_x(t) \frac{20r\sin^2(t/2)dt}{((1-r)^2 + 4r\sin^2(t/2))^3} \le$$

$$\le \frac{2}{(1-r)^4}\int_0^{\pi(1-r)} t\bar{w}_x(t)dt +$$

$$+ \frac{10r}{(1-r)^6}\int_0^{\pi(1-r)} t^3\bar{w}_x(t)dt \le 26\pi^2 \frac{\bar{w}_x(\pi(1-r))}{(1-r)^2}.$$

For J_2, we use the estimate

$$R_r'(t) \le \frac{21\sin^2(t/2)}{((1-r)^2+ 4r\sin^2(t/2))^3} \le \frac{21}{64r^3\sin^4(t/2)} \qquad (\pi(1-r)\le t\le \pi).$$

Hence, we get

$$|J_2| \le 6\pi^4\int_{\pi(1-r)}^{\pi}\frac{\bar{w}_x(t)}{t^3}dt .$$

In view of the inequality

$$(1-r)^2 \int_{\pi(1-r)}^{\pi} \frac{\bar{w}_x(t)}{t^3}\,dt \geq \frac{3}{8\pi^2}\,\bar{w}_x(\pi(1-r)),$$

$$Y_1 \leq \frac{1}{\pi}\left|h(r)-\frac{1+r}{2}\right|(1-r)^2(1+r)r(|J_1|+|J_2|) \leq$$

$$\leq \left(\frac{80\pi^3}{3}+12\pi^3\right)\left|h(r)-\frac{1+r}{2}\right|(1-r)^2 \int_{\pi(1-r)}^{\pi} \frac{\bar{w}_x(t)}{t^3}\,dt \quad ,$$

and this completes the proof.

3. Estimates for the Weierstrass integral. It is known (see [1] pp.201-204, 125-126) that if $f \in L$, then the operator (3) can be written in the form

$$W_r[f](x) = \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{+\infty} f(x-t)e^{-t^2/4\alpha}\,dt \ ,$$

where $\alpha = \log\frac{1}{r}$ $(0<r<1)$.

THEOREM 3. Let x be a weak Lebesgue point (4) of f and let $r\in\langle\frac{1}{2},1)$. Then

(9) $$|W_r[f](x)-f(x)| \leq 4\alpha\sqrt{\alpha}\,w_x(\pi;f) + \frac{6}{5\alpha\sqrt{\alpha}} \int_{\pi}^{\pi} t^2 w_x(t;f)e^{-t^2/4\alpha}\,dt.$$

Proof. Putting $\varphi_x(t) = f(x+t)+f(x-t)-2f(x)$, we easily verify that

$$W_r[f](x)-f(x) = \frac{1}{2\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \int_{2\pi j}^{2\pi(j+1)} \varphi_x(t)e^{-t^2/4\alpha}\,dt =$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \int_0^{2\pi} \varphi_x(2\pi j+t)e^{-(2\pi j+t)^2/4\alpha}\,dt =$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \int_0^{\pi} \varphi_x(t)G_j(t)\,dt \quad ,$$

where $G_j(t) = e^{-(2\pi j+t)^2/4\alpha} + e^{-(2\pi(j+1)-t)^2/4\alpha}$.

Integrating by parts we get

$$\sum_{j=0}^{\infty} \int_0^{\pi} \varphi_x(t)G_j(t)\,dt = \sum_{j=0}^{\infty}\left\{\Phi_x(\pi)G_j(\pi) - \int_0^{\pi} \Phi_x(t)G_j'(t)\,dt\right\} .$$

where $\Phi_x(t)=\int_0^t \varphi_x(u)du$. Therefore, in view of the inequality

$$|\Phi_x(t)| \le 2tw_x(t),$$

we have

$$|W_r[f](x)-f(x)| \le \frac{1}{\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \pi w_x(\pi) G_j(\pi) +$$

$$+ \frac{1}{\sqrt{\pi\alpha}} \int_0^{\pi} tw_x(t)\,|G_0'(t)|\,dt + \frac{1}{\sqrt{\pi\alpha}} \sum_{j=1}^{\infty} \int_0^{\pi} tw_x(t)\,|G_j'(t)|\,dt =$$

$$= M + N + S, \text{ say.}$$

Clearly,

$$G_j(\pi) = 2e^{-((2j+1)\pi)^2/4\alpha} \le \frac{64\alpha^2}{\pi^4} \frac{1}{(2j+1)^4} \quad (j=0,1,2,\ldots)$$

and

$$M \le \frac{64\alpha\sqrt{\alpha}}{\pi^3\sqrt{\pi}} w_x(\pi) \sum_{j=0}^{\infty} \left(\frac{1}{2j+1}\right)^4 = \frac{2\sqrt{\pi}}{3} \alpha \sqrt{\alpha}\, w_x(\pi).$$

It is easy to see, that

$$G_j'(t) = \frac{1}{2\alpha} \Big\{ (2\pi(j+1)-t)e^{-(2\pi(j+1)-t)^2/4\alpha} -$$

$$- (t+2\pi j)e^{-(t+2\pi j)^2/4\alpha} \Big\} \qquad (j=0,1,2,\ldots).$$

Therefore

$$N = \frac{1}{\sqrt{\pi\alpha}} \int_0^{\pi} tw_x(t)\,|G_0'(t)|\,dt \le \frac{1}{2\alpha\sqrt{\pi\alpha}} \int_0^{\pi} t^2 w_x(t) e^{-t^2/4\alpha}\,dt +$$

$$+ \frac{1}{2\alpha\sqrt{\pi\alpha}} \int_0^{\pi} (2\pi-t)e^{-(2\pi-t)^2/4\alpha}\, tw_x(t)\,dt \le$$

$$\le \frac{1}{2\alpha\sqrt{\pi\alpha}} \int_0^{\pi} t^2 w_x(t) e^{-t^2/4\alpha}\,dt + \frac{\pi\sqrt{\pi}}{\alpha\sqrt{\alpha}} w_x(\pi) \int_0^{\pi} e^{-(2\pi-t)^2/4\alpha}\,dt \le$$

$$\le \frac{1}{2\alpha\sqrt{\pi\alpha}} w_x(\sqrt{\alpha}) \int_0^{\sqrt{\alpha}} t^2 e^{-t^2/4\alpha}\,dt + \frac{1}{2\alpha\sqrt{\pi\alpha}} \int_{\sqrt{\alpha}}^{\pi} t^2 w_x(t) e^{-t^2/4\alpha}\,dt +$$

$$+ 384\pi\sqrt{\pi}\alpha\sqrt{\alpha}\, w_x(\pi) \int_{\pi}^{2\pi} \frac{1}{u^5}\,du \le$$

$$\leq \frac{2}{\sqrt{\pi}} w_x(\sqrt{\alpha}) + \frac{1}{2\alpha\sqrt{\pi\alpha}} \int_{\sqrt{\alpha}}^{\pi} t^2 w_x(t) e^{-t^2/4\alpha} dt + \frac{77\alpha\sqrt{\alpha}}{\pi^3\sqrt{\pi}} w_x(\pi).$$

Further, if $j \neq 0$ and $t \in \langle 0, \pi \rangle$,

$$|G_j'(t)| \leq \frac{1}{2\alpha} e^{-(2\pi j+t)^2/4\alpha} \{2\pi(j+1)-t+2\pi j+t\} =$$

$$= \frac{\pi}{\alpha}(2j+1)e^{-(2\pi j+t)^2/4\alpha} \leq \frac{\pi}{\alpha}(2j+1)e^{-\pi^2 j^2/\alpha} e^{-t^2/4\alpha} \leq$$

$$\leq \frac{2\alpha(2j+1)}{\pi^3 j^4} e^{-t^2/4\alpha} \leq \frac{6\alpha}{\pi^3 j^3} e^{-t^2/4\alpha} \leq \frac{6\alpha}{\pi^3 j^2} e^{-t^2/4\alpha} .$$

Consequently,

$$S = \frac{1}{\sqrt{\pi\alpha}} \sum_{j=1}^{\infty} \int_0^{\pi} t w_x(t) |G_j'(t)| dt \leq \frac{6\sqrt{\alpha}}{\pi^3\sqrt{\pi}} \sum_{j=1}^{\infty} (\frac{1}{j})^2 \int_0^{\pi} t w_x(t) e^{-t^2/4\alpha} dt \leq$$

$$\leq \frac{\sqrt{\alpha}}{\pi\sqrt{\pi}} w_x(\sqrt{\alpha}) \int_0^{\sqrt{\alpha}} t e^{-t^2/4\alpha} dt + \frac{\sqrt{\alpha}}{\pi\sqrt{\pi}} \int_{\sqrt{\alpha}}^{\pi} t w_x(t) e^{-t^2/4\alpha} dt \leq$$

$$\leq \frac{\alpha\sqrt{\alpha}}{2\pi\sqrt{\pi}} w_x(\sqrt{\alpha}) + \frac{1}{\pi\sqrt{\pi}} \int_{\sqrt{\alpha}}^{\pi} t^2 w_x(t) e^{-t^2/4\alpha} dt .$$

Collecting the above results and using the estimate

$$w_x(\sqrt{\alpha}) \leq \frac{1}{10\alpha\sqrt{\alpha}} \int_{\sqrt{\alpha}}^{\pi} t^2 w_x(t) e^{-t^2/4\alpha} dt$$

we obtain our thesis.

REMARK 2. The estimate (9) cannot be essentialy improved. To prove this, let us consider the class $L(\Omega)$. In view of (9),

$$\sup_{f \in L(\Omega)} |W_r[f](x) - f(x)| \leq 4\alpha\sqrt{\alpha}\,\Omega(\pi) + \frac{6}{5\alpha\sqrt{\alpha}} \int_{\sqrt{\alpha}}^{\pi} t^2 \Omega(t) e^{-t^2/4\alpha} dt .$$

Considering the non-negative function $g \in L(\Omega)$, defined by the identities $g(t) = \Omega(|t-x|)$ if $|t-x| \leq \pi$, $g(t+2\pi) = g(t)$ we shall show that the last supremum can be estimated similarly from below.

Namely,

$$\sup_{f\in L(\Omega)} |W_r[f](x)-f(x)| \geq |W_r[g](x)-g(x)| = \frac{1}{\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \int_0^{\pi} \Omega(t) G_j(t)\,dt \geq$$

$$\geq \frac{1}{\sqrt{\pi\alpha}} \int_0^{\pi} \Omega(t) G_0(t)\,dt \geq \frac{1}{\sqrt{\pi\alpha}} \int_{\sqrt{\alpha}}^{\pi} \Omega(t) e^{-t^2/4\alpha}\,dt \geq$$

$$\geq \frac{1}{4\alpha\sqrt{\pi\alpha}} \int_{\sqrt{\alpha}}^{\pi} t^2 \Omega(t) e^{-t^2/2\alpha}\,dt .$$

REMARK 3. In view of the inequality $e^{-u} \leq 6u^{-3}$ $(u > 0)$, the estimate (9) yields

$$|W_r[f](x)-f(x)| \leq 4\alpha\sqrt{\alpha}\, w_x(\pi;f) + 461\alpha\sqrt{\alpha} \int_{\sqrt{\alpha}}^{\pi} \frac{w_x(t;f)}{t^4}\,dt .$$

Replacing α by $\log \frac{1}{r}$ and applying the inequality

$$1-r \leq \log \frac{1}{r} \leq 2(1-r) \quad (\tfrac{1}{2} \leq r < 1),$$

we conclude that

$$|W_r[f](x)-f(x)| \leq 12(1-r)\sqrt{1-r}\, w_x(\pi;f) + 922\sqrt{2}(1-r)\sqrt{1-r} \int_{\sqrt{1-r}}^{\pi} \frac{w_x(t;f)}{t^4}\,dt$$

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PUNKTOWA APROKSYMACJA FUNKCJI CAŁKAMI TYPU ABELA

Streszczenie

Dla danej 2π-okresowej funkcji f całkowalnej w sensie Lebesgue'a badane są pewne średnie jej szeregu Fouriera i trygonometrycznego szeregu sprzężonego. Szacowany jest rząd zbieżności tych średnich w punktach Lebesgue'a funkcji f. Otrzymane twierdzenia uzupełniają wyniki Starka (1972) i Hsiang Wen-Hanga (1982).