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On compact stochastic perturbations of mappings of the unit interval

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Let φ_n be continuous mappins of a compact metric space X and $\Re$ be some fixed compact Markov operator on $C(X)$. We study the asymptotic ($n \longrightarrow \infty$) behavior of invariant probability measure of the compositions $(T_{\varphi_n} R)^*$, where $\varphi_n \longrightarrow \varphi$ uniformly. We apply our general results to the investigation of the difference equation $X_{n+1} = \varphi(X_n) + W$, where W is a fixed random variable independent of n and X_n and φ are continuous maps from $[0,1]$ into $[0,1]$.

It is shown that for a wide class of mappings φ this Markov process admits the unique (stationary) invariant measure $\mu(\varphi)$ and the mapping $\varphi \longrightarrow \mu(\varphi)$ is continuous.

Let (X, ρ) be a compact metric space. We denote by $C(X)$ the Banach lattice of all continuous functions on X, and by $P(X)$ the set of all Borel probability measures on X. The smallest, closed set of all full measure $\mu \in P(X)$ is denoted by supp μ (the support of μ). A linear operator $T : C(X) \longrightarrow C(X)$ is said to be Markov if $T1 = 1$, and $f \geq 0 \Longrightarrow Tf \geq 0$. It is well known that for every Markov operator T on $C(X)$ there exits a unique family of probability measures $P(x, \cdot)$, on X such that

(a) for every Borel set A the mapping $x \longrightarrow P(x, A)$ is Borel measurable

(b) for every $f \in C(X)$, $Tf(x) = \int f(y)P(x, dy)$.

In fact, we have $P(x,\cdot) = T^*\delta_x(\cdot)$. By $B(X)$ we denote the set of bounded Borel functions on X. Using property (b) the Markov operator T can be canonically extended to an operator $T : B(X) \longrightarrow B(X)$. We say that a closed set $A \subset X$ is T–invariant (or simply, invariant if T is fixed) if for every $x \in A$ we have $P(x, A) = 1$ (equivalentely $T1_A \geq 1_A$).

Now, let Ω be the product space $\prod_{n=0}^{\infty} X_n$ where $X_n = X$. We equip Ω with its natural topology and product σ–field. Let η_n be the natural projection $\eta_n : \Omega \longrightarrow X_n$. It is well known (e.g. [3] Proposition 2.10 p.18) that given any initial distribution probability μ on X there is a probability $\mathcal{P}_\mu$ defined on Ω with

$$\mathcal{P}_\mu\{\eta_k \in A_k : k = 0, 1, \ldots, n\} = \int 1_{A_0} T(1_{A_1} \ldots 1_{A_{n-1}}(T1_{A_n}) \ldots) d\mu.$$

Moreover, the sequence $\{\eta_k\}_{k=0}^{\infty}$ is a homogenous Markov chain with transition probability $P(\cdot,\cdot)$ and starting measure μ. It is called the canonical Markov chain with transition probability P.

Let $P_T(X)$ denote the set of all T^*–invariant measures. Clearly, it is nonempty, convex and w^*–compact subset of $P(X)$, and for every μ belonging to $P_T(X)$ the canonical Markov chain $\{\eta_k\}$ is stationary with respect to $\mathcal{P}_\mu$.

In the sequel we will need some informations on compact Markov operators. Recall that a linear operator $T : C(X) \longrightarrow C(X)$ is compact if and only if the mapping $x \longrightarrow T^*\delta_x$ is norm continuous, so every compact Markov operator is strong Feller ($TB(X) \subset C(X)$ see [3] Proposition 5.8 p.37). The Cesaro means $A_n f = n^{-1}(f + Tf + \ldots + T^{n-1}f)$ of a compact Markov operator T converge uniformly to a finite-dimensional projection, ex $P_T(X)$ (it means the set of extremal measures) is finite and $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset$ for each distinct extremal T^*–invariant measures μ_1, μ_2 (see [2] and [4] for detailes). Observe that every continuous mapping $\varphi : X \longrightarrow X$ defines a Markov (deterministic) operator $T_\varphi f = f \circ \varphi$.

Now let M denote the convex, semitopological semigroup of all Markov operators on $C(X)$. For an operator $R \in N$ the mapping $M \ni T \longrightarrow T \circ R$ is denoted by $\mathcal{R}$. Observe that if T_φ is a deterministic Markov operator induced by some transformation φ then $\mathcal{R}(T_\varphi)f(x) = T_\varphi \circ Rf(x) = Rf(\varphi(x)) = \int f(y)R(\varphi(x), dy)$. If for every $x \in X$ the transition measure $R(x,\cdot)$ is concentrated on the ball

$B(x,r) = \{y \in X : \rho(x,y) \leq r\}$, then our operator $\mathcal{R}(T_\varphi)$ can be recognized as an r-perturbation of the dynamical system (X,φ). In the sequel we will consider only compact perturbations (thus $\mathcal{R}(T)$ is a compact for every $T \in M$). Clearly, $\mathcal{R}(T_n) \xrightarrow{\text{s.o.t.}} \mathcal{R}(T)$ whenever $T_n \xrightarrow{\text{s.o.t.}} T$ (here s.o.t. means strong operator topology). Using the compactness of the operator R we get the following.

Lemma 1 *Let the sequence (T_n) of Markov operators on $C(X)$ converge in the strong operator topology to T and R be a fixed compact Markov operator on $C(X)$. Then $\mathcal{R}(T_n) \longrightarrow \mathcal{R}(T)$ in the operator norm and every limit measure $\mu = w^* - \lim_{j \longrightarrow \infty} \mu_{n_j}$ is $\mathcal{R}(T)^*$ invariant (here $\mu_n \in P_{\mathcal{R}(T_n)}(X)$). Moreover, in this case $\| \mu - \mu_{n_j} \| \longrightarrow 0$, where $\| \cdot \|$ is the variation.*

Proof. By the compactness of R we can chose a finite, ε-dense subset $f_1, \ldots, f_m$ of $R(K_1)$, where K_1 denotes the unit ball of $C(X)$. Since $T_n \xrightarrow{\text{s.o.t.}} T$ thus there exists n_0 such that for $n \geq n_0$ we have $\| T_n f_j - T f_j \| \leq \varepsilon$ for every $j = 1, \ldots, m$. So, for every $f \in C(X)$, $\| f \| \leq 1$ we get

$$\| \mathcal{R}(T_n)f - \mathcal{R}(T)f \| \leq \| \mathcal{R}(T_n)f - \mathcal{R}(T_n)f_j \| +$$
$$+ \| \mathcal{R}(T_n)f_j - \mathcal{R}(T)f_j \| + \| \mathcal{R}(T)f_j - \mathcal{R}(T)f \| \leq 3\varepsilon$$

if the index j is suitable. Thus,

$$\| \mathcal{R}(T_n) - \mathcal{R}(T) \| = \sup_{\|f\| \leq 1} \| \mathcal{R}(T_n)f - \mathcal{R}(T)f \|$$

tends to 0.

Now assume that $\mu = w^* - \lim_{j \longrightarrow \infty} \mu_{n_j}$ for some $\mu_{n_j} \in P_{\mathcal{R}(T_{n_j})}(X)$. For every $f \in C(X)$ we have

$$\left| \int f \, d\mu - \int \mathcal{R}(T)f \, d\mu \right| \leq \left| \int f \, d\mu - \int f \, d\mu_{n_j} \right| +$$

$$\left| \int \mathcal{R}(T)f \, d\mu_{n_j} - \int \mathcal{R}(T_{n_j})f \, d\mu_{n_j} \right| + \left| \int \mathcal{R}(T)f \, d\mu_{n_j} - \int \mathcal{R}(T)f \, d\mu \right|.$$

Since the components of the right side of the previous inequality converge to 0, thus μ is $\mathcal{R}(T)^*$ invariant probability. Next

$$\| \mu - \mu_{n_j} \| = \sup_{\|f\| \leq 1} \left| \int f \, d\mu - \int f \, d\mu_{n_j} \right. =$$

$$\sup_{\|f\|\leq 1} | \int \mathcal{R}(T)f\,d\mu - \int \mathcal{R}(T_{n_j})f\,d\mu_{n_j} | \leq$$

$$\sup_{\|f\|\leq 1} | \int \mathcal{R}(T)f\,d\mu - \int \mathcal{R}(T)f\,d\mu_{n_j} | +$$

$$| \int \mathcal{R}(T)f\,d\mu_{n_j} - \int \mathcal{R}(T_{n_j})f\,d\mu_{n_j} | \leq$$

$$\sup_{\|f\|\leq 1} | \int \mathcal{R}(T)f\,d\mu - \int \mathcal{R}(T)f\,d\mu_{n_j} | + \| \mathcal{R}(T) - \mathcal{R}(T_{n_j}) \| \leq$$

$$\sup_{\|f\|\leq 1} \{ | \int TRf\,d\mu - \int Tf_i d\mu | + | \int Tf_i\,d\mu - \int Tf_i d\mu_{n_j} | +$$

$$| \int Tf_i\,d\mu_{n_j} - \int \mathcal{R}(T)Tf d\mu_{n_j} | \} + \| \mathcal{R}(T) - \mathcal{R}(T_{n_j}) \| \leq$$

$$\sup_{1\leq i\leq n} | \int Tf_i\,d\mu - \int Tf_i d\mu_{n_j} | + \| \mathcal{R}(T) - \mathcal{R}(T_{n_j}) \| + 2\varepsilon$$

(here $\{f_1, \ldots, f_m\}$ is an ε–dense subset of $R(K_1)$). Since ε can be taken arbitrarily small, then $\| \mu - \mu_{n_j} \| \longrightarrow 0$.

The following corollary strengthens some results from [1].

Corollary 1 *If the Markov operator $\mathcal{R}(T)$ has exactly one invariant probability measure μ (i.e. it is uniquely ergodic) and $T_n \xrightarrow{\text{s.o.t.}} T$ then $\| \mu_n - \mu \| \to 0$ (here $\mu_n \in P_{\mathcal{R}(T_n)}(X)$). In particular there exists n_0 such that for $n \geq n_0$ the operators $\mathcal{R}(T_n)$ are uniquely ergodic.*

Now, let φ be a continuous mapping from the unit interval [0,1] into itself. Consider the stochastic difference equation $X_{n+1} = \varphi(X_n) + W$, where W is a small random variable possessing the probability density function $g : [-a, a] \longrightarrow [0, \infty)$, where a is small and $g \geq 0$, $\int_{-a}^{a} g\,d\lambda = 1$ for the Lebesgue measure λ (i.e. $Prob.(W \in A) = \int_A g(x)dx$ for every Borel set $A \subset [-a, a]$). We assume that the perturbation term W is independent of n and X_n.

We say that a closed subset ([1] Definition 1) $S \subset [0, 1]$ is invariant with respect to our stochastic difference equation if $\varphi(S) \oplus [-a, a] \subset S$, where $\oplus$ is defined $A \oplus B = \{x + y : \; x \in A, y \in B\}$. We will say S is (φ, a) invariant then. Let us observe that every (φ, a) invariant subset S is invariant in an ordinary sense (i.e. $\varphi(S) \subset S$). If there exists a (φ, a) invariant set S then clearly X_n is a Markov process with state space S. More precisely, for every initial probability μ concentrated on S there exists the Markov process X_n such that

$$\mathcal{P}_\mu(X_{n+1} - \varphi(X_n) \in A) = \int_A g \, d\lambda$$

for every Borel $A \subset [-a, a]$. We show that the process X_n defined on some (φ, a) invariant subset can be regarded as a part of the Markov perturbation of the dynamical system $([0,1], \varphi)$. We set

$$(c) \ \frac{d\, R^*\delta_x}{d\,\lambda}(t) = \begin{cases} g(t-x), \text{ for } x \in [a, 1-a] \\ a^{-2}(a-x) \cdot 1_{[0,a)}(t) + a^{-1} \cdot x \cdot g(t-a), \text{for } x \in [0, a) \\ a^{-2}(x-(1-a))1_{(1-a,1]}(t) + \\ \qquad a^{-1}(1-x) \cdot g(t-(1-a)), \text{ for } x \in (1-a, 1] \end{cases}$$

where λ denotes the Lebesgue measure. Clearly, the mapping

$$x \longrightarrow R^*\delta_x$$

is norm continuous, so the Markov operator R given by the above transition function is compact.

Moreover, since $R^*\delta_x \prec\!\prec \lambda$ then $R^*(P(S)) \subset L^1(S, \lambda)$ and it is easy to see that

$$\frac{d\, R^*\mu}{d\,\lambda}(t) = \int \frac{d\, R^*\delta_x}{d\,\lambda}(t)\, \mu(dx) \quad \text{for every } \mu \in P(S).$$

We notice that if g is a function of bounded variation then similarly $\frac{d\, R^*\delta_x}{d\,\lambda}$ has bounded variation too, and the following rough estimation

$$Var(\frac{d\, R^*\delta_x}{d\,\lambda}) \leq Var(g) + 2a^{-1}$$

holds.

Consider the perturbation of T_φ by the Markov operator R and let η_n be the canonical Markov process defined by $\mathcal{R}(T_\varphi)$. For every Borel set $A \subset S$ and $x \in S$ (notice that $\varphi(x) \in [a, 1-a]$ then) we have

$$P(\eta_{n+1} \in A \mid \eta_n = x) = \mathcal{R}(T_\varphi)1_A(x) = (R \circ T_\varphi)^*\delta_x(A) =$$

$$= R^*\delta_{\varphi(x)}(A) = \int_A g(y - \varphi(x))\, d\lambda(y).$$

On the other hand if $x \in S$ then

$$P(X_{n+1} \in A \mid X_n = x) = P(\varphi(X_n) + W \in A \mid X_n = x) =$$

$$= \int_{A-\varphi(x)} g(y)\, d\lambda(y) = \int_A g(y - \varphi(x))\, d\lambda(y).$$

Since X_n and η_n are Markov processes we conclude that all finite dimensional distributions coincide. So on the phase subspace S they give equivalent descriptions of the perturbation of the dynamical system $([0,1], \varphi)$.

Assume that φ is nonsingular (i.e. if $\lambda(A) = 0$ for a Borel set $A \subset [0,1]$ then $\lambda(\varphi^{-1}(A)) = 0$). The Frobenius-Perron operator connected with φ is an operator defined on $L^1([0,1])$ as follows

$$P_\varphi f(x) = \frac{d}{dx} \int_{\varphi^{-1}[0,x)} f(y)\, dy.$$

In [1] Boyarski has studied the stochastic operators Q_φ defined on $L^1(S)$ by the equation $Q_\varphi f(x) = (P_\varphi f * g)(x)$, where $*$ denotes the convolution and g is the probability density function of W. The properties of invariant densities have been investigated there. In particular it was shown that the "regularity" of g implies some nice properties of invariant density. Since for a (φ, a) invariant set S the operators Q_φ and $\mathcal{R}(T_\varphi)^*$ coincide on $L^1(S)$ (notice that S is an $\mathcal{R}(T_\varphi)$ invariant subset then), thus $\{f \in L^1(S) : \ Q_\varphi f = f\} = P_{\mathcal{R}(T_\varphi)}(S)$. The existence of an absolutely continuous invariant measure of Q_φ is a simple consequence of the last equality. Moreover it does exist for arbitrary continuous mapping (not necessary nonsingular or C^1) from [0,1] into itself.

Corollary 2 *If for a continuous mapping $\varphi : [0,1] \longrightarrow [0,1]$ there exists a closed (φ, a) invariant set S then for every random variable W with absolutely continuous density function $g : [-a, a] \longrightarrow [0, \infty)$ the stochastic process defined by $X_{n+1} = \varphi(X_n) + W$ has a stationary probability distribution of the form $\mathcal{P}_f$ for some positive, normalized $f \in L^1(S, \lambda)$. In particular for every Borel set $A \subset S$ $\mathcal{P}_f(X_n \in A) = \int_A f\, d\lambda$.*

Now, let us fix the probability density function g of W and assume that supp $g \subset [-a, a]$. Our next result corresponds to Theorem 1 from [1].

Proposition 1 *Let φ be a continuous mapping from $[0,1]$ into itself and $\mathcal{R}(T_\varphi)$ be the compact perturbation of T_φ (here R is defined by (c)). If g is of bounded variation then the density of each $\mathcal{R}(T_\varphi)^*$ invariant probability μ is of bounded variation too.*

Proof. Let $\mu \in P_{\mathcal{R}(T_\varphi)}([0,1])$ be arbitrary. Since

$$\frac{d\,\mu}{d\,\lambda}(t) = \int \frac{d\,\mathcal{R}(T_\varphi)^*\delta_x}{d\,\lambda}(t)\;\mu(dx),$$

then for every $0 = t_0 < t_1 < \ldots < t_n = 1$ we get

$$\sum_{i=1}^{m} \left| \frac{d\mu}{d\lambda}(t_i) - \frac{d\mu}{d\lambda}(t_{i-1}) \right| \leq$$

$$\sum_{i=1}^{m} \left| \frac{d\,\mathcal{R}(T_\varphi)^*\delta_x}{d\,\lambda}(t_i) - \frac{d\,\mathcal{R}(T_\varphi)^*\delta_x}{d\,\lambda}(t_{i-1}) \right| d\mu(x) \leq Var(g) + 2a^{-1} < \infty.$$

The following theorem connects the smothness of invariant densities with properties of g.

Theorem 1 *Let φ be a continuous mapping from $[0,1]$ into itself and S be (φ, a) invariant subset of $[0,1]$.*

If φ is nonsingular and $g \in L^\infty([-a,a])$ then density of every $\mathcal{R}(T_\varphi)^*$–invariant probability μ supported on S is a continuous function.

If $g \in C_0^k([-a,a])$ (i.e. $g(-a) = g(a) = 0$ and g has continuous k derivatives) then $\frac{d\mu}{d\lambda} \in C^k([0,1])$ for each invariant probability $\mu \in P_{\mathcal{R}(T_\varphi)}(S)$.

Proof. Let φ be nonsingular and $\mu \in P_{\mathcal{R}(T_\varphi)}(S)$. Since for each $t \in [0,1]$

$$\frac{d\,\mu}{d\,\lambda}(t) = \int_S g(t - \varphi(x))d\mu(x) = \int_S g(t-x)d\mu\circ\varphi^{-1}(x) =$$

$$= \int g(t-x)\frac{d\mu\circ\varphi^{-1}}{d\lambda}(x)d\lambda(x) = g \star \frac{d\mu\circ\varphi^{-1}}{d\lambda}(t)$$

then $\frac{d\mu}{d\lambda} \in C([0,1])$ with supp $\frac{d\mu}{d\lambda} \subset S$. Now, let $g \in C_0^k([0,1])$. Then

$$\frac{\frac{d\,\mu}{d\,\lambda}(t+h) - \frac{d\,\mu}{d\,\lambda}(t)}{h} =$$

$$= \lim_{h \longrightarrow \infty} \frac{\int g(t+h-\varphi(x)) - g(t-\varphi(x))\, d\mu(x)}{h} = \int g'(t-\varphi(x))\, d\mu(x),$$

so $\frac{d\mu}{d\lambda} \in C^1([0,1])$. Similarly we get higher derivatives.

Remark In the previous theorem the nonsingularity assumption on φ is essential. In fact let $\varphi(x) = 2^{-1}$ for all $x \in [0,1]$ and

$$g(x) = \begin{cases} (4a)^{-1} & \text{for} \quad x \in [-a, 0), \\ 3(4a)^{-1} & \text{for} \quad x \in [0, a]. \end{cases}$$

where $a < 4^{-1}$ is fixed. Clearly, the unit interval is (φ, a) invariant and the (unique) $\mathcal{R}(T_\varphi)$ invariant probability μ has the density of bounded variation. But it can be computed that

$$\frac{d\,\mu}{d\,\lambda}(t) = \int g(t-\varphi(x))d\mu(x) =$$

$$g(t-2^{-1}) = \begin{cases} (4a)^{-1} & \text{for} \quad t \in [2^{-1}-a, 2^{-1}), \\ 3(4a)^{-1} & \text{for} \quad t \in [2^{-1}, 2^{-1}+a], \end{cases}$$

and $\frac{d\mu}{d\lambda} \notin C([0,1])$.

Proposition 2 *Let φ be a nonsingular continuous mapping from $[0,1]$ into itself and $g \in L^1([-a,a])$ be a density of some perturbation of φ such that $0 \in \operatorname{supp} g$.*

If there are no two disjoint non-meager φ–invariant subsets of $[0,1]$ then $\mathcal{R}(T_\varphi)$ is uniquely ergodic.

Proof. As in the first part of our theorem 1, if μ is $\mathcal{R}(T_\varphi)^*$–invariant probability, then

$$\frac{d\,\mu}{d\,\lambda}(t) = \int g(t-x)\frac{d\mu \circ \varphi^{-1}}{d\lambda}(x)d\lambda(x)$$

and thus

$$\{t \in [0,1] \; : \; \frac{d\,\mu}{d\,\lambda}(t) > 0\}$$

has nonempty interior. But the topological support of every invariant probability is an invariant subset of [0,1] (see [4] for detailes). So, because $0 \in \operatorname{supp}\, g$, we get that the support of every $\mathcal{R}(T_\varphi)^*$ invariant probability is some φ–invariant subset with nonempty interior.

Since supports of distinct extremal invariant probabilities of a compact Markow operator are disjoint (see [4]) thus because of the assumptions of our theorem the operator $\mathcal{R}(T_\varphi)$ must be uniquely ergodic.

Recall that a continuous mapping φ from [0,1] into itself is said to be transitive if there exists a point $x_0 \in [0,1]$ such that the orbit $\{\varphi^n(x_0)\}_{n\geq 0}$ is dense in [0,1]. The following result is a simple consequence of the previous results.

Corollary 3 *Let φ be a continuous, nonsingular and transitive mapping from $[0,1]$ into itself and $g \in L^1([-a,a])$ be the probability density function such that $0 \in \operatorname{supp} g$. Then the Markov operator $\mathcal{R}(T_\varphi)$ is uniquely ergodic.*

The following theorem expresses the continuous dependence of the invariant measure of the uniquely ergodic compact perturbation $\mathcal{R}(T_\varphi)$, on the mapping φ.

Theorem 2 *Let $\varphi_n \longrightarrow \varphi$ uniformly on $[0,1]$ where φ_n, φ are continuous mappings from $[0,1]$ into itself. If for some positive a there exists a closed (nonempty) (φ,a)–invariant set S and there are no two non-meager disjoint φ–invariant sets then for every probability density function $g \in L^1_+((-\infty,+\infty))$ satisfying $0 \in \operatorname{supp} g \subset [-a,a]$, we have*

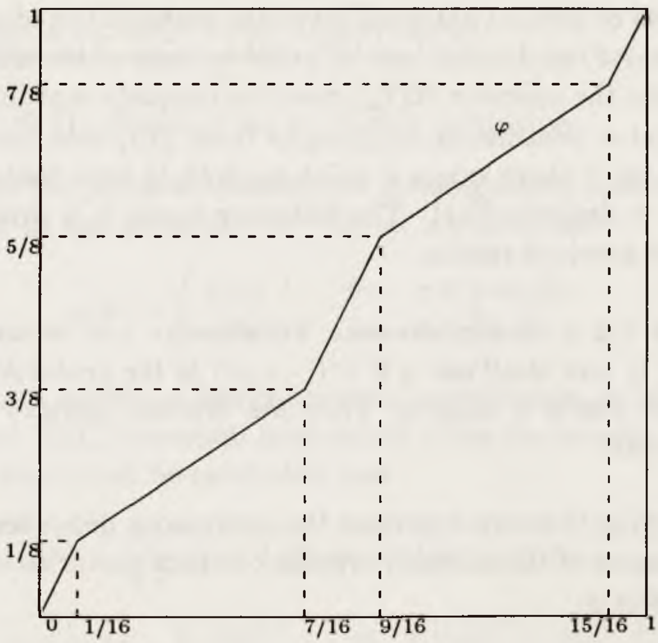
$$\| \mu_n - \mu \| = \int_0^1 \left| \frac{d\mu_n}{d\lambda} - \frac{d\mu}{d\lambda} \right| d\lambda \longrightarrow 0$$

where μ_n, μ are $\mathcal{R}(T_{\varphi_n})^*$, $\mathcal{R}(T_\varphi)^*$ invariant probabilities, respectively.

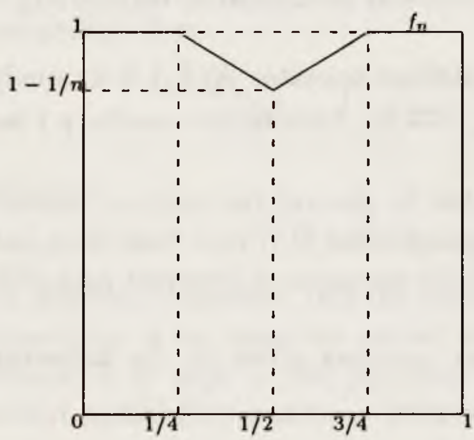
Proof. By our proposition 2 the Markov operator $\mathcal{R}(T_\varphi)$ is uniquely ergodic. Since $\varphi_n \longrightarrow \varphi$ implies $T_{\varphi_n} \overset{\text{s.o.t.}}{\longrightarrow} T_\varphi$, thus by our corollary 1 we get the thesis.

The following example shows that in general the uniform convergence $\varphi_n \longrightarrow \varphi$ of continuous mappings from [0,1] into itself does not guarantee the convergence of a suitable sequence of invariant probabilities.

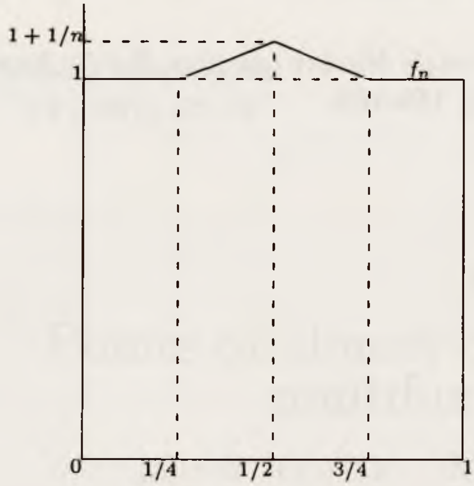
Example Let φ be the continuous mapping given by the following diagram:



Consider the density $g = 8$ on $[-(16)^{-1}, (16)^{-1}]$ and let R be the appropriate compact Markov operator defined as in (c). Clearly $A = [(16)^{-1}, 7(16)^{-1}]$, $B = [9(16)^{-1}, 15(16)^{-1}]$ are the only $(\varphi, (16)^{-1})$ invariant subsets of [0,1]. Thus $P_{\mathcal{R}(T_\varphi)}([0,1])$ has exactly two extremal measures concentrated on A and B respectively. Now, we define the sequence of continuous functions f_n on the unit interval:



for odd n



and consider the sequence $\varphi_n = f_n\varphi$ (clearly $\varphi_n \longrightarrow \varphi$). Observe that for odd n the mappings φ_n have exactly one $(\varphi_n, (16)^{-1})$ invariant subset and it is contained in $[(16)^{-1}, 7(16)^{-1}]$. For even n the mappings φ_n have also exactly one $(\varphi_n, (16)^{-1})$ invariant subset, but it is contained in $[9(16)^{-1}, 15(16)^{-1}]$. Thus, for every natural n, we have $\| \mu_n - \mu_{n+1} \| = 2x$ where $\mu_j \in P_{\mathcal{R}(T_{\varphi_j})}([0,1])$ and the sequence of measures μ_n does not converge.

Remark It is easy to observe that by a small modifications in the previous example the mappings φ, φ_n can be taken to be smooth.

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