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Points of almost continuity of real multifunctions

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In this article we consider the local property of almost continuity at a point of real multifunctions. The following question is interesting: Is it possible to define this local property and moreover to make sure that this property characterize the global almost continuity of a multifunction? We give such a characterization in this article.

Throughout the article we shall make use of the following notions and notations.

Consider the following topological properties that a multifunction F from one topological space X to another one Y may have:

$\mathcal{C}$: F is continuous,

$\mathcal{D}$: F is a Darboux function iff for every connected subset C of X, the set $F(C)$ is a connected subset of Y.

$\mathcal{AC}$: F is almost continuous function (in the sense of Stallings [5]) iff for every open set $G \subseteq X \times Y$ containing F there exists a continuous function $g : X \longrightarrow Y$ such that $g \subseteq G$.

One can see that if a multifunction F is $\mathcal{AC}$ then for every G, for every $x \in (a, b)$ and every $y \in F(x)$ there is a continuous function g such that $g \subseteq G$ and $g(x) = y$.

The multifunctions we consider are real multifunctions defined on an interval (open or closed) with connected values. Recall that for those multifunctions the following implications hold:

$$\mathcal{C} \Rightarrow \mathcal{AC} \Rightarrow \mathcal{D}.$$

By $L(F,x)$, $L^+(F,x)$, $L^-(F,x)$ we shall denote the set of all limit points, the set of all right-hand sided limit points, the set of all left-hand sided limit points of the multifunction F at the point x, respectively.

Definition 1 *A multifunction $F : (a,b) \longrightarrow \mathbb{R}$ is said to be almost continuous at a point $x \in (a,b)$ from the right-hand side if*

(i) $F(x) \cap L^+(F,x) \neq \emptyset$,

(ii) there is a positive ε such that for arbitrary neighbourhood G of $F|[x,\infty)$, arbitrary $y \in (\lim_{t \longrightarrow x^+} \inf F(t), \lim_{t \longrightarrow x^+} \sup F(t))$, arbitrary neighbourhood U of the point (x,y) and each $t \in (x, x+\varepsilon)$ there exists a continuous function $g : [x, x+\varepsilon] \longrightarrow \mathbb{R}$ such that $g \subseteq G \cup U$, $g(x) = y$ and $g(t) \in F(t)$.

In the analogous manner we define a multifunction that is almost continuous at a point from the left-hand side. A multifunction is called almost continuous at a point iff it is almost continuous at that point from both sides. For a multifunction $F : [a,b] \longrightarrow \mathbb{R}$ we can say that it is almost continuous at the points a and b if it is so from the right-hand side at a and from the left-hand side at b.

One can easily observe that if a multifunction is continuous at a point from any side then it is almost continuous at that point from the same side.

Property 1 *If $F : (a,b) \longrightarrow \mathbb{R}$ is almost continuous at a point x_0, then there is a positive ε such that for an arbitrary neighbourhood G of $F|[x,\infty)$, arbitrary $y \in (\lim_{t \longrightarrow x^+} \inf F(t), \lim_{t \longrightarrow x^+} \sup F(t))$, arbitrary neighbourhood U of the point (x,y), arbitrary $t \in (x, x+\varepsilon)$ and arbitrary $z \in F(t)$ there exists a continuous function $g_1 : [x, x+\varepsilon] \longrightarrow \mathbb{R}$ such that $g_1 \subseteq G \cup U$, $g_1(x) = y$ and $g_1(t) \in F(t)$.*

Theorem 1 *If $F : (a,b) \longrightarrow \mathbb{R}$ is almost continuous, then it is almost continuous at every point of the interval (a,b).*

Proof If F is continuous at a point x_0 from the right-hand side then, of course, it is almost continuous at that point from the right-hand side.

If not, then $L^+(F, x_0)$ is a nondegenerate interval. Let y be an arbitrary point from the interval

$$(\inf L^+(F, x_0), \sup L^+(F, x_0))$$

and G be an arbitrary neighbourhood of $F|[x_0, b)$, U—arbitrary neighbourhood of (x_0, y) and t—a point from (x_0, b). There is a point of F contained in U, let it be (x_1, y_1) where $y_1 \in F(x_1)$. We can take that $x_1 > x_0$. There is a function $g_1 : [x_0, x_1] \longrightarrow \mathbb{R}$ such that

(a) g_1 is continuous, $g_1 \subseteq U$,

(b) $g_1(x_0) = y$ and $g_1(x_1) = y_1$.

The multifunction F is almost continuous in (a, b), then for its neighbourhood G and the point t from $[x_1, b)$ there is a function $g_2 : [x_1, b) \rightarrow \mathbb{R}$ such that

(a') g_2 is continuous, $g_2 \subseteq G$,

(b') $g_2(x_1) \in F(x_1)$ and $g_2(t) \in F(t)$.

In this way the funcion $g : [x_0, b) \longrightarrow \mathbb{R}$ given by

$$g(x) = \begin{cases} g_1(x) & \text{for } x \in [x_0, x_1], \\ g_2(x) & \text{for } x \in (x_1, b), \end{cases}$$

is continuous and $g \subseteq G \cup U$. This proves that F is almost continuous at x_0 from the right-hand side.

Similarly one can prove that F is also almost continuous at x_0 from the left-hand side.

Theorem 2 *If a multifunction $F : [a, b] \longrightarrow \mathbb{R}$ is almost continuous at every point of the interval $[a, b]$, then F is almost continuous in $[a, b]$.*

Proof Let G be any neighbourhood of F. Let $x \in [a, b]$ and $z \in F(x)$. Consider two possibilities:

– x is a point of right-hand side continuity of F,

– x is not such point.

In each possibility there is $\delta_1 > 0$ such that for every $t_1 \in (x, x+\delta_1)$ there exists a continuous function $g_{x,\delta_1,t_1} : [x, x+\delta_1] \longrightarrow \mathbb{R}$ fulfilling all the conditions:

(α_1) $g_{x,\delta_1,t_1}(x) = z, \quad g_{x,\delta_1,t_1}(t_1) = z_1,$

(β_1) $g_{x,\delta_1,t_1} \subseteq G.$

Similarly there is $\delta_2 > 0$ such that for every $t_2 \in (x-\delta_2, x)$ there exists a continuous function $g_{x,\delta_2,t_2} : [x-\delta_2, x] \longrightarrow \mathbb{R}$ fulfilling all the conditions:

(α_2) $g_{x,\delta_2,t_2}(x) = z, \quad g_{x,\delta_2,t_2}(t_2) = z_2,$

(β_2) $g_{x,\delta_2,t_2} \subseteq G.$

Hence for every $x \in [a,b]$ there is a positive ε_x such that for every t_1, $t_2 \in (x-\varepsilon_x, x+\varepsilon_x)$ for which $t_1 < x < t_2$ there exists a continuous function $g_{x,\varepsilon_x,t_{1,2}} : [x-\varepsilon_x, x+\varepsilon_x] \longrightarrow \mathbb{R}$ fulfilling all the conditions

(α) $g_{x,\varepsilon_x,t_1,t_2}(x) = z, \quad g_{x,\varepsilon_x,t_1,t_2}(t_i) = z_i$ for $i = 1, 2,$

(β) $g_{x,\varepsilon_x,t_1,t_2} \subseteq G.$

The family $\{(x-\varepsilon_x, x+\varepsilon_x) : x \in [a,b]\}$ is a cover of the interval $[a,b]$, so there exists a finite sequence of points $(x_1, \dots, x_n)$ such that

$$x_1 < x_2 < \dots < x_n \text{ and } [a,b] = \bigcup_{i=1}^{n} (x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i}) .$$

We can assume that each interval $(x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i})$ has common points only with the preceding and subsequent intervals of that sequence.

Now let $t_i \in (x_i, x_i + \varepsilon_{x_i}) \cap (x_{i-1} - \varepsilon_{x_{i-1}}, x_{i-1})$ and $t_0 = a$, $t_n = b$ and $z_i \in F(t_i)$ for $i = 1, \dots, n$. Put $g(x) = g_{x_i,\varepsilon_{x_i},t_{i-1},t_i}(x)$ for $x \in [t_{i-1}, t_i]$, $i = 1, \dots, n$. Then the function $g : [a,b] \longrightarrow \mathbb{R}$ is continuous and of course, $g \subseteq G$, what proves that F is almost continuous in $[a,b]$.

It is obvious that if $F : (a, a_0] \longrightarrow \mathbb{R}$ has the property that $F|[a_n, a_0]$ is almost continuous, where $a_n \in (a, a_0)$ and $a_n \longrightarrow a$, then F is almost continuous in $(a, a_0]$.

Now we are ready to state the following theorem.

Theorem 3 *If $F : (a,b) \longrightarrow \mathbb{R}$ is almost continuous at every point of the interval (a,b), then it is almost continuous in (a,b).*

For the points of almost continuity of a multifunction we can prove the analogue of the theorem on asymmetry.

Theorem 4 *The set of all points of the interval (a,b), at which a multifunction $F : (a,b) \longrightarrow \mathbb{R}$ is almost continuous from exactly one side is at most countable.*

Proof Let A be the set of asymmetry of almost continuity of a multifunction F. Let us denote by B the set of all points, at which F is almost continuous from the right-hand side and is not almost continuous from the left-hand side. By C we denote the set $A \setminus B$. Let

$$D_1 = \{x \in (a,b) : L^+(F,x) \neq L^-(F,x)\},$$

$$D_2 = \{x \in (a,b) : F(x) \cap L^+(F,x) \cap L^-(F,x) \neq \emptyset\},$$
$$E = B \setminus (D_1 \cup D_2).$$

The sets D_1, D_2 are countable [3]. We shall show that the set E is also countable. Let E_n be the set of all points $x_0 \in E$ such that the diameter of the set $L(F,x_0)$ is greater than or equal to $\frac{1}{n}$ and such that for every neighbourhood G of $F|[x_0, x_0 + \frac{1}{n}]$, every $t \in (x_0, x_0 + \frac{1}{n}]$ and $z \in F(x)$, each y from the interval $(\inf L^+(F,x_0), \sup L^+(F,x_0))$ and every neighbourhood U of (x_0, y) there exists a continuous function $g : [x_0, x_0 + \frac{1}{n}] \longrightarrow \mathbb{R}$ such that $g \subseteq G \cup U$, $g(x_0) = y$ and $g(t) = z$.
Now let (x_k) be an increasing sequence of points from E_n converging to x_0. We shall show that $x_0 \notin E_n$. Since $L(F, x_k)$, for $k \in \mathbb{N}$ has the diameter greater than or equal to $\frac{1}{n}$, then so has $L^-(F,x_0)$. There is x_k such that $0 < x_0 - x_k < \frac{1}{n}$.
Let $\varepsilon = x_0 - x_k$, $y \in (\inf L^-(F,x_0), \sup L^-(F,x_0))$ and U be a neighbourhood of (x_0, y), t belong to (x_k, x_0) and $z \in F(t)$. We can assume that

$$U = (x_0 - \delta, x_0 + \delta) \times (y - \delta, y + \delta),$$
$$t < x_0 - \delta, \text{ and}$$
$$(y - \delta, y + \delta) \subseteq (\inf L^-(F,x_0), \sup L^-(F,x_0)).$$

For δ there are points x', x'' such that

$$x' < x'',\ x'' \in (x_0 - \delta, x_0)$$

and

$$F(x') \cap (y + \delta, +\infty) \neq \emptyset,\ F(x'') \cap (-\infty, y - \delta) \neq \emptyset.$$

Let G be an arbitrary neighbourhood of $F|[x_k, x_0]$ and

$$H = \{(t, z) : z \in F(t)\}.$$

Of course there is an open square $U_1 \times U_2$ such that $H \subset U_1 \times U_2 \subset G$. Put

$$G' = G \setminus (t \times (\mathbb{R} \setminus U_2)).$$

Consider two possibilities:

- $F(x') \cap (-\infty, y + \delta) \neq \emptyset$. Since $\{(x', z') : z \in F(x')\}$ is connected, then there is $z' \in F(x')$ such that $(x', z') \in U$. Now let $G'' = G' \cup U$, $\bar{x} = x'$. Then there exists a continuous function

$$g : [x_k, x_k + \frac{1}{n}] \longrightarrow \mathbb{R}$$

such that $g \subseteq G''$, $g(\bar{x}) = z'$.

- $F(x') \cap (-\infty, y + \delta) = \emptyset$. Put $G'' = (G' \cup U) \setminus (\{x'\} \times (-\infty, y + \delta])$. Of course, G'' is a neighbourhood of $F|[x_k, x_0]$ and $x'' - x_k < \frac{1}{n}$ hence there is a continuous function $g : [x_k, x_k + \frac{1}{n}] \longrightarrow \mathbb{R}$ such that

$$g \subseteq G'', g(x_k) \in F(x_k) \text{ and } g(x'') = z''$$

where $z'' \in F(x'') \cap (-\infty, y - \delta)$. The function g, as a continuous one, cuts the square U and let $\bar{x}$ be a point from $(x_0 - \delta, x_0)$ for which $g(\bar{x}) \in (y - \delta, y + \delta)$. Then the function $h : [x_k, x_0] \longrightarrow \mathbb{R}$ given by

$$h(x) = \begin{cases} g(x) & \text{for } x \in [x_k, \bar{x}] \setminus U_1 \\ z & \text{for } x = t \\ \text{linear} & \text{in each of the intervals } \overline{U_1},\ [\bar{x}, x_0] \end{cases}$$

is continuous and contained in $G \cup U$. This means that F is almost continuous at x_0 from the left-hand side, thus $x_0 \notin E_n$.

Hence the set E_n contains no left-hand sided accumulation point of E_n, so it is countable.

Similarly, C is countable and so is A.

Theorem 5 *The set of all points of almost continuity of an arbitrary multifunction is of type G_δ.*

Proof Let $F : (a,b) \longrightarrow \mathbb{R}$ be an arbitrary multifunction. By $\mathcal{A}^+(F)$ ($\mathcal{A}^-(F)$) we shall denote the set of all points of (a,b) at which F is almost continuous from the right-hand side (left-hand side), and $\mathcal{A}(F) = \mathcal{A}^+(F) \cap \mathcal{A}^-(F)$. Let A_n be the set of those points of $\mathcal{A}(F)$ for which the ε from Definition 1 is greater than $\frac{1}{n}$. We shall show that

$$A_n \subseteq \mathrm{Int}\,(A_n \cup \mathcal{C}(F)),$$

here $\mathcal{C}(F)$ denotes the set of all points of continuity of the function F.

Let $x_0 \in A_n$ and $\delta = \varepsilon - \frac{1}{n}$. We shall prove that

$$(x_0 - \delta, x_0 + \delta) \subseteq A_n \cup \mathcal{C}(F).$$

If $x \in (x_0 - \delta, x_0 + \delta) \setminus \mathcal{C}(F)$ and, for example, $x_0 < x$, G is an arbitrary neighbourhood of F, $y \in (\inf L(F,x), \sup L(F,x))$ and U is a square neighbourhood of (x,y), then there are t', t'' such that

$$t' < x < t'', (t', y') \in U \text{ and } (t'', y'') \in U$$

for $y' \in F(t')$, $y'' \in F(t'')$.

Let $W = W_1 \times W_2 \subset G$ be a neighbourhood of $\{t'\} \times F(t')$ and $V = V_1 \times V_2 \subset G$ — a neighbourhood of $\{t''\} \times F(t'')$. Then

$$G' = (G \cup U) \setminus (([t' \times (\mathbb{R} \setminus \mathrm{U}_2)]) \cup ([t'' \times (\mathbb{R} \setminus \mathrm{V}_2)]))$$

is a neighbourhood of $F|[x_0, b]$ and there exists a continuous function g such that $g(x_0) \in F(x_0)$ and $g(t) \in F(t)$. The function g cuts the square W and V, hence there is a continuous function $h : [x, x_0 + \varepsilon) \longrightarrow \mathbb{R}$ such that $h(x) = y$, $h(t) \in F(t)$ and $h \subseteq G' \subseteq G \cup U$.

Similarly, for $t \in (x - \varepsilon, x)$ there exists a continuous function $h' : [x - \varepsilon, x] \longrightarrow \mathbb{R}$ such that $h' \subseteq G \cup U$, $h'(x) = y$ and $h'(t) \in F(t)$.

Thus if $x_0 \in A_n$, then $x_0 \in \mathrm{Int}\,(A_n \cup \mathcal{C}(F))$.

Since $\mathcal{A}(F) = \bigcup_{n=1}^{\infty} A_n \cup \mathcal{C}(F)$ and

$$\bigcup_{n=1}^{\infty} A_n \cup \mathcal{C}(F) = \bigcup_{n=1}^{\infty} \operatorname{Int}\,(A_n \cup \mathcal{C}(F)) \cup \mathcal{C}(F),$$

then $\mathcal{A}(F)$ is of the type G_δ.

In the end of the article it is worth mentioning that Theorem 5 gives an exact characterization of the set of all points of almost continuity of a multifunction.

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