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Density Topologies

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$\mathcal{I}$–density topology, which can be considered as a category analogue of the density topology, has been introduced in [6]. Basic facts concerning properties of this topology have been collected in [9]. Recently the second author in [11] introduced a new kind of lower density involving simultaneously measure and category. In this note we shall discuss superpositions of three lower density operators, two of them being "category" lower densities and the third "measure" lower density. We shall show that in this way it is possible to obtain only three different "category" lower densities, two of which were described in [2] and the third in [11].

Let X be an arbitrary non-empty set, $\mathcal{B}$—a σ–algebra of subsets of X and $\mathcal{I}$—a proper σ–ideal included in $\mathcal{B}$. We shall write $A \sim_{\mathcal{I}} B$ for A, $B \subset X$ to indicate that $A \triangle B \in \mathcal{I}$. Following [4], p. 88 and [6] we shall say that a mapping $F : \mathcal{B} \longrightarrow \mathcal{B}$ is a lower density for $(\mathcal{B}, \mathcal{I})$ if and only if the following conditions are fulfilled:

LD.1. For each $A \in \mathcal{B}$ $\quad F(A) \sim_{\mathcal{I}} A$.

LD.2. If $A, B \in \mathcal{B}$ and $A \sim_{\mathcal{I}} B$, then $F(A) = F(B)$.

LD.3. $F(\emptyset) = \emptyset$, $\;F(X) = X$.

LD.4. For each $A, B \in \mathcal{B}$ $\quad F(A \cap B) = F(A) \cap F(B)$.

Observe that from LD.1. and LD.2. we have immediately

LD.5. For each $A \in \mathcal{B}$ $\quad F(F(A)) = F(A)$.

In the sequel we shall consider the case when $X = \mathbb{R}$ (the real line), $\mathcal{B}$ is the family of sets having the Baire property and $\mathcal{I}$ is the σ–ideal of sets of the first category. In this situation we shall say that $F : \mathcal{B} \longrightarrow \mathcal{B}$ is a category lower density if it is a lower density for $(\mathcal{B}, \mathcal{I})$ and fulfills the following condition:

LD.6. If $A \in \mathcal{B}$, and so $A = G \triangle P$, where G is regularly open and $P \in \mathcal{I}$ (this representation is unique), then $G \subset F(A) \subset \overline{G}$ ($\overline{G}$ denotes the closure of G in the natural topology).

Here are definitions of two announced category lower densities. Let

$$F_r(A) = G \text{ for open} A = G \triangle P,$$

where G is regularly open and $P \in \mathcal{I}$ (see [2] and [4], p. 87–88). We shall call F_r a regular lower density. It is easily seen that F_r is the smallest one among all category lower densities.

To define the next category lower density we shall need some denotations (see [6], [9]): if $A \subset \mathbb{R}$ and $t \in \mathbb{R}$, then $t \cdot A = \{t \cdot x : x \in A\}$ and $A + t = \{x + t : x \in A\}$; χ_A denotes the characteristic function of A. We shall say that some property holds $\mathcal{I}$–almost everywhere ($\mathcal{I}$–a.e.) if and only if the set of points which do not have this property belongs to $\mathcal{I}$. Now we say that 0 is an $\mathcal{I}$–density point of a set $A \in \mathcal{B}$ if and only if for each increasing sequence $\{n_m\}_{m \in N}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in N}$ such that the sequence $\left\{\chi_{(n_{m_p} \cdot A) \cap [-1,1]}\right\}_{p \in N}$ converges to $\chi_{[-1,1]}$ $\mathcal{I}$–a.e. We shall say that x_0 is an $\mathcal{I}$–density point of $A \in \mathcal{B}$ if and only if 0 is an $\mathcal{I}$–density point of $A - x_0$. Let $F_{\mathcal{I}}(A) = \{x : x \text{ is an } \mathcal{I}\text{–density point of } A\}$ for $A \in \mathcal{B}$. We shall call $F_{\mathcal{I}}$ an $\mathcal{I}$–lower density.

In [2] it is proved that for each $A \in \mathcal{B}$ $F_{\mathcal{I}}(A)$ is a Borel set (in fact, it is $F_{\sigma\delta}$). Obviously $F_r(A)$, being open, is a Borel set.

Now recall "measure" lower density. To do this let $\mathcal{S}$ be a σ–algebra of all Lebesgue measurable sets on the real line and $\mathcal{L}$—a σ–ideal of null sets. Let $F_d(A) = \{x : \lim_{h \to 0+} [(2h)^{-1} \cdot m(A \cap [x - h, x + h])] = 1\}$ for $A \in \mathcal{S}$, where m stands for linear Lebesgue measure. It is well known (see, for example, [4]; p. 18) that F_d is a lower density for $(\mathcal{S}, \mathcal{L})$. Also it is known (see [8]) that for each $A \in \mathcal{S}$, $F_d(A)$ is a Borel set, namely, it is $F_{\sigma\delta}$.

We are ready for studying superpositions. Let $F_{dr} : \mathcal{B} \longrightarrow \mathcal{B}$ be a mapping defined by $F_{dr} = F_d \circ F_r$. The denotations $F_{d\mathcal{I}}, F_{\mathcal{I}r}, F_{r\mathcal{I}}$ are self-explaining. Since we are interested in category lower densities rather than in measure, we shall not study $F_{\mathcal{I}d}$ and F_{rd} (see remark 1 at the end of the note). All operators $F_{dr}, F_{d\mathcal{I}}, F_{\mathcal{I}r}, F_{r\mathcal{I}}$ are well defined, because the inner values are Borel sets. We need not study F_{rr} and $F_{\mathcal{I}\mathcal{I}}$ by virtue of LD.5.

Proposition 1 *$F_{\mathcal{I}r} = F_{\mathcal{I}}$, $F_{r\mathcal{I}} = F_r$, $F_{d\mathcal{I}}$ is a category lower density and $F_{d\mathcal{I}} = F_{dr}$.*

Proof. Let $A \in \mathcal{B}$. We have $F_r(A) \sim_{\mathcal{I}} A$ by LD.1. for F_r and next $F_{\mathcal{I}r}(A) = F_{\mathcal{I}}(A)$ by LD.2. for $F_{\mathcal{I}}$. The proof of the second equality is the same.

Since LD.2.–LD.4. are preserved by superpositions, we shall prove LD.6. for $F_{d\mathcal{I}}$ and then LD.1. will follow immediately. To prove LD.6. observe that for each open set G the following inclusions hold:

$$G \subset F_d(G) \subset \overline{G}.$$

Hence $F_{d\mathcal{I}}$ is a category lower density. To prove the third equality we shall need the following lemma:

Lemma 1 *If G is regularly open, then $m(F_{\mathcal{I}}(G) \triangle G) = 0$.*

Proof. Since $G \subset F_{\mathcal{I}}(G)$ we have to prove only that

$$m(F_{\mathcal{I}}(G) \setminus G) = 0.$$

But $F_{\mathcal{I}}(G) \subset \overline{G} = G \cup \partial G$ (here ∂G denotes the boundary of G), so $F_{\mathcal{I}}(G) \setminus G \subset \partial G$. The lemma will be proved if we show that

$$\partial G \cap F_d(\partial G) \subset \partial G \setminus F_{\mathcal{I}}(G),$$

since then $F_{\mathcal{I}}(G) \setminus G \subset \partial G \cap F_{\mathcal{I}}(G) \subset \partial G \setminus F_d(\partial G)$ and the last set has measure zero by virtue of the classical Lebesgue theorem.

Let $x_0 \in \partial G \cap F_d(\partial G)$. For the simplicity of estimations suppose that $x_0 = 0$ (if not, use the translation). Since 0 is a measure density point of ∂G, then from the observation in [6] (resulting among others is the definition of $\mathcal{I}$–density point) it follows that for each increasing sequence $\{n_m\}_{m\in N}$ of natural numbers there exists a subsequence $\left\{n_{m_p}\right\}_{p\in N}$ such that the sequence $\left\{\chi_{(n_{m_p}\cdot\partial G)\cap[-1,1]}\right\}_{p\in N}$ converges almost everywhere (i.e. $\mathcal{L}$–a.e.) to $\chi_{[-1,1]}$. It means that

$$m([-1,1]\setminus\liminf_p(n_{m_p}\cdot\partial G))=0.$$

Hence

$$[-1,1]\cap\liminf_p(n_{m_p}\cdot\partial G)$$

is dense on $[-1,1]$ and obviously $[-1,1]\cap\limsup_p(n_{m_p}\cdot\partial G)$ is also dense on $[-1,1]$. It follows that for each $\{n_m\}_{m\in N}$ the set

$$[-1,1]\cap\limsup_m(n_m\cdot\partial G)$$

is dense on $[-1,1]$. Observe that if x is an arbitrary point in ∂G, then each neighbourhood of x intersects open set $\mathbb{R}\setminus\overline{G}$, since G is regularly open. Now

$$\limsup_m(n_m\cdot\partial G)=\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}(n_k\cdot\partial G),$$

so for each $m\in N$ the set $\bigcup_{k=m}^{\infty}(n_k\cdot\partial G)$ is dense in $[-1,1]$. From the above observation it follows immediately that for each $m\in N$ the set $\bigcup_{k=m}^{\infty}(n_k\cdot(\mathbb{R}\setminus\overline{G}))$ is open and dense in $[-1,1]$, so $\limsup_m(n_m\cdot(\mathbb{R}\setminus\overline{G}))$ is residual in $[-1,1]$ for each increasing sequence $\{n_m\}_{m\in N}$. Suppose now that 0 is an $\mathcal{I}$–density point of G (and so of $\overline{G}$). Then from the definition it follows immediately that for some subsequence $\{n_m\}_{m\in M}$ the set $\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty}(n_m\cdot\overline{G})\cap[-1,1]=[-1,1]\cap\liminf_m(n_m\cdot\overline{G})$ is residual in $[-1,1]$—a contradiction. So 0 cannot be an $\mathcal{I}$–density point G and $0\in\partial G\setminus F_I(G)$. The lemma is proved.

Now we shall return to the proof of the proposition.

Let $A\in\mathcal{B}$, $A=G\triangle P$ as usual. Since $F_{\mathcal{I}}(A)=F_{\mathcal{I}}(G)$ by LD.2 for $F_{\mathcal{I}}$, we have $m(F_{\mathcal{I}}(A)\triangle F_r(A))=m(F_I(G)\triangle G)=0$ from the above lemma. Then $F_d(F_{\mathcal{I}}(A))=F_d(F_r(A))$ by LD.2 for F_d.

Following [11] we shall use the denotation F_c for F_{dr}. So using "double" superpositions we have obtained three different category lower densities. Indeed, if $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ are two decreasing sequences of real numbers tending to zero such that $a_{n+1} < b_{n+1} < a_n$ for $n \in N$, $\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = 0$ and $\lim_{n\to\infty} \frac{a_n - b_{n+1}}{a_n} = 1$, then

$$G = \mathbb{R} \setminus (\{0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n])$$

is a regularly open set such that $0 \in F_{\mathcal{I}}(G)$ and $0 \in F_d(G)$
but obviously $0 \notin F_r(G) = G$. Hence $F_{\mathcal{I}} \neq F_r$, $F_d = F_r$ and $F_c = F_r$ (compare [9], th. 2 and [7]).

Further, if $\{d_n\}_{n\in N}$ is a decreasing sequence tending sufficiently quickly to zero, then

$$G = \mathbb{R} \setminus \left(\{0\} \cup \bigcup_{n=1}^{\infty} [n^{-1}, n^{-1} + d_n] \right)$$

is a regularly open set such that $0 \in F_d(G)$ but $0 \notin F_{\mathcal{I}}(G)$. Hence $F_d \neq F_{\mathcal{I}}$ and $F_c \neq F_{\mathcal{I}}$. Modifying the example from th.1.(d) in [10] we can show that there exists a regular open set G such that $0 \in F_{\mathcal{I}}(G)$ but $0 \notin F_d(G)$. Hence neither $F_d \subset F_{\mathcal{I}}$, $F_c \subset F_{\mathcal{I}}$ nor $F_{\mathcal{I}} \subset F_d$, $F_{\mathcal{I}} \subset F_c$. In [11] it was shown
that we have also neither $F_d \subset F_c$,$F_r \subset F_c$ nor $F_c \subset F_d$, $F_c \subset F_r$.

Now we proceed to "triple" superpositions. We shall use again self-explaining denotations such as $F_{d\mathcal{I}r}$ etc.

Proposition 2 $F_{d\mathcal{I}r} = F_c$, $F_{dr\mathcal{I}} = F_c$, $F_{\mathcal{I}dr} = F_{\mathcal{I}}$, $F_{rdr} = F_r$, $F_{rd\mathcal{I}} = F_r$, $F_{\mathcal{I}d\mathcal{I}} = F_{\mathcal{I}}$, $F_{r\mathcal{I}r} = F_r$, $F_{\mathcal{I}r\mathcal{I}} = F_{\mathcal{I}}$.

Proof. All proofs are very simple and similar. As an example we shall prove the first and the third equality. $F_{d\mathcal{I}r} = F_d \circ F_{\mathcal{I}r} = F_d \circ F_{\mathcal{I}} = F_{d\mathcal{I}} = F_c$ from proposition 1. Since F_{dr} is a lower density for $(\mathcal{B}, \mathcal{I})$, we have $F_{dr}(A) \sim_{\mathcal{I}} A$ for $A \in \mathcal{B}$. Hence $F_{\mathcal{I}}(F_{dr}(A)) = F_{\mathcal{I}}(A)$ by LD.2. for $F_{\mathcal{I}}$.

Theorem 1 *If $\{a_1, a_2, \ldots, a_n\}$ is any sequence of symbols r, $\mathcal{I}$, d such that $a_n \neq d$, then $F_{a_1 a_2 \ldots a_n} = F_{a_1} \circ F_{a_2} \circ \ldots \circ F_{a_n}$ is a category lower density and equals either to F_r, either to $F_{\mathcal{I}}$ or to F_c.*

Proof. Superposition is associative, so if any two consecutive terms in the above sequence are equal we can replace them by a single term using

LD.5. and according to proposition 2 every three-term sequence with r or $\mathcal{I}$ on the third place can be replaced by single term or two-term sequence dr. Using repeatedly these two reductions finaly we obtain $F_{\mathcal{I}}$, F_r or $F_{dr} = F_c$.

Remark 1 We didn't deal with F_{rd} and $F_{\mathcal{I}d}$, because F_d is a lower density for $(\mathcal{S}, \mathcal{L})$ and not for $(\mathcal{B}, \mathcal{I})$. Moreover, $F_{\mathcal{I}d}$ is not a lower density for $(\mathcal{S}, \mathcal{L})$, because LD.1. is not fulfilled (for $\mathcal{L}$, of course). Indeed, let $C \subset [0,1]$ be a Cantor set of positive measure. Put $A = (0,1) \setminus C$. Then A is an open set dense in (0,1), so $F_d(A) \supset A$ is residual in (0,1). Hence $F_{\mathcal{I}d}(A) = (0,1)$ and $F_{\mathcal{I}d}(A) \setminus A = C \notin \mathcal{L}$. One can suppose that if we restrict F_d to $\mathcal{B} \cap \mathcal{S}$, then $F_{\mathcal{I}d}$ is a lower density for $(\mathcal{B} \cap \mathcal{S}, \mathcal{I})$. It is not the case, as the following example shows.

Let $A \subset (0,1)$ be a first category Borel set such that $m(A) = 1$. Then $F_d(A) = (0,1)$, so $F_{\mathcal{I}d}(A) = (0,1)$. Hence $F_{\mathcal{I}d}(A) \setminus A \notin \mathcal{I}$. The same examples may be applied to F_{rd}.

It is well known that if F is a lower density for $(\mathcal{B}, \mathcal{I})$, then

$$T = \{F(A) \setminus P; A \in \mathcal{B}, P \in \mathcal{I}\}$$

is a topology and $T \subset \mathcal{B}$ if $(\mathcal{B}, \mathcal{I})$ fulfils countable chain conditions. If we use lower densities $F_{\mathcal{I}}$, F_r and F_c to construct topologies, then we obtain $T_{\mathcal{I}}$, T^* and T_c, which are described in details in [6], [2] and [11], respectively.

Remark 2 Let us notice here some simple facts on a.e. modifications of T_r, $(T_{\mathcal{I}}$, T_c topologies, respectively.

The a.e. modification of T_r–topology is the same topology.

The a.e. modification of $T_{\mathcal{I}}$–topology is introduced and examined in [5] and denoted there by $T_{\mathcal{I}}^1$.

The a.e. modification of T_c–topology we shall denote similarly by T_c^1. It is not difficult to establish the following inclusions:

$$T_c \supseteq T_c^1 \supseteq \text{a.e.–topology}.$$

In fact. Let C be a Cantor set of positive measure in [0,1]. Let $x_0 \in F_d(C)$. The set $([0,1] \setminus C) - x_0$ is T_c^1–open but not a.e.–open. On the other side the set $[0,1] \setminus Q$, where Q is the set of all rational numbers, is T_c–open but not T_c^1–open.

Finally we shall present some result on functions continuous with respect to topology generated by composition of lower densities. We shall formulate it in more general form. Let us recall first some facts on semiregular spaces. Let (X, T) be a topological space. The family of all T–regular open sets $RO(X, T)$ forms a base for a smaller topology T_s on X, called the semiregularization of T. The space (X, T) is said to be semiregular if $T_s = T$.

Proposition 3 *For any topological space (X, T), $(T_s)_s = T_s$.*

Proposition 4 *Let f be a function on X with value in regular space. If f is T–continuous than it is T_s–continuous.*

See [3] for references and [1, pages 200–201, th 1.13, 1.14] for simple proofs.

Now let $\mathcal{B}_i$, i=1,2 be two σ–algebras of subsets of nonempty set X, $\mathcal{I}_i \subset \mathcal{B}_i$ proper σ–ideals and $F_i : \mathcal{B}_i \longrightarrow \mathcal{B}_i$ lower densities for $(\mathcal{B}_i, \mathcal{I}_i)$. We shall assume additionally that $(\mathcal{B}_i, \mathcal{I}_i)$ fulfils countable chain condition and superposition $F_1 \circ F_2$ denoted as F_{12} is a lower density for $(\mathcal{B}_2, \mathcal{I}_2)$. We can introduce two topologies in X :

$$T_{F_1} = \{F_1(A) \setminus P : \quad A \in \mathcal{B}_1,\ P \in \mathcal{I}_1\},$$

$$T_{F_{12}} = \{F_{12}(A) \setminus P : \quad A \in \mathcal{B}_2,\ P \in \mathcal{I}_2\}.$$

The proofs of the following three lemmas can be done similarly like proofs of theorems 22.6, 22.7, 22.8 in [4].

Lemma 2 *The set $P \subset X$ is nowhere dense in (X, T_{F_1}) (in $(X, T_{F_{12}})$) if and only if $P \in \mathcal{I}_1$ ($P \in \mathcal{I}_2$). Every nowhere dense set is closed.*

Lemma 3 *The set $A \subset X$ has the Baire property in (X, T_{F_1}) (in $(X, T_{F_{12}})$) if and only if $A \in \mathcal{B}_1$ $(A \in \mathcal{B}_2)$.*

Lemma 4 *The set $G \subset X$ is regularly open in (X, T_{F_1}) (in $(X, T_{F_{12}})$) if and only if $G = F_1(A)$ for some $A \in \mathcal{B}_1$ ($G = F_{12}(A)$ for some $A \in \mathcal{B}_2$).*

By $C(X, T)$ we denote the class of T–continuous functions on X with values in regular topological space.

Theorem 2 *We have* $C(X, T_{F_{12}}) \subset C(X, T_{F_1})$.

Proof. From lemma 3 $RO(X, T_{F_{12}}) \subset RO(X, T_{F_1})$ hence $(T_{F_{12}})_s$ is finer than $(T_{F_1})_s$, and consequently, $C(X, (T_{F_{12}})_s) \subset C(X, (T_{F_1})_s)$. As from proposition 2 $C(X, T_{F_{12}}) = C(X, (T_{F_{12}})_s)$ and $C(X, T_{F_1}) = C(X, (T_{F_1})_s)$ we have $C(X, T_{F_{12}}) \subset C(X, T_{F_1})$.

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