Problemy Matematyczne
13 (1992), 33–41

On Some Subclasses of $\mathcal{DB}_1$ Functions

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1. Introduction. Studying the behavior of real functions H.Rosen [10] defined *Baire .5* functions and showed that every function *Baire .5* with Darboux property is quasi–continuous. It is easy to see, that for every function f, if f is a *Baire .5* function then f is in the first class of Baire and there exist functions which are in the first class and not *Baire .5*. We shall see below (in Example 1.1) that there exists Darboux *Baire 1* function which is quasi-continuous but is not *Baire .5*.

Let us establish some terminology to be used later. $\mathbb{R}$ denotes the set of all reals. A functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *quasi-continuous* at a point $x \in \mathbb{R}$ if for every open neighbourhoods U of x and V of $f(x)$ there exists a non-empty open set $W \subset U \cap f^{-1}(V)$. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be quasi-continuous on $\mathbb{R}$ iff it is quasi-continuous at each point $x \in \mathbb{R}$.
A set A is said to be *semi-open* if $A \subset \overline{\mathrm{Int}A}$ — [7] (by $\overline{A}$ and $\mathrm{Int}A$ we denote the closure and the interior of A). It is well-known that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is quasi-continuous iff for every non-empty open set $V \subset \mathbb{R}$ the set $f^{-1}(V)$ is semi-open. $\mathcal{Q}$ denotes the family of all real quasi-continuous functions. Moreover we shall consider the following families of real functions defined on $\mathbb{R}$.

lsc (*usc*) — the class of all lower (upper) semi-continuous functions,

$\mathcal{C}$ — the class of all continuous functions,

$\mathcal{D}$ — the class of all Darboux functions, i.e. the class of all function f for which $f(C)$ is connected whenever C is connected.

$\mathcal{B}_1$ — the class of all functions of the first class of Baire,

$\mathcal{B}.5$ — the class of all functions f such that for every open set $U \subset \mathbb{R}$ the set $f^{-1}(U)$ is a G_δ set,

$\mathcal{M}$ $[\mathcal{M}_0]$ — the class of all functions f which satisfy the following condition: if x_0 is a right (left) hand sided point of discontinuity of f, then $f(x_0) = 0$ and there exists a sequence $\{x_n\}_{n=1}^{\infty}$ [of points of continuity of f] such that $f(x_n) = 0$ for every $n \in \mathcal{N}$ and $x_n \searrow x_0 (x_n \nearrow x_0)$ [3],

$\mathcal{Y}$ — the family of all functions with the Young property, i.e. functions which are bilaterally dense in themselves (some authors call functions having this property peripherally continuous),

$\mathcal{A}$ — the family of all *almost continuous functions* in the sense of Stallings, i.e. functions f such that the for every open subset of $\mathbb{R}^2$ containing f contains a continuous function (no difference between a function and its graph is made),

$\mathcal{C}onn$ — the class of all connectivity functions, i.e. functions f such that $f|C$ is a connected subset of $\mathbb{R}^2$ whenever C is a connected subset of $\mathbb{R}$.

It is well-known that the following inclusions hold: $\mathcal{M} \subset \mathcal{DB}_1$ and $usc \subset \mathcal{B}_1 \supset lsc$. Moreover, J. Young showed in [11] that for *Baire 1* functions, Darboux and Young properties are equivalent. K. Kuratowski, W. Sierpiński and J. Brown [6], [1] proved that for functions of the first class notions of almost continuity, connectednes and Darboux property are equivalent.

T. Natkaniec showed in [9] that a function is quasi-continuous and satisfies the Young condition iff for every point $x_0 \in \mathbb{R}$ there exist two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ of continuity points of f such that $x_n \searrow x_0, z_n \nearrow x_0$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n) = f(x_0)$.

In this paper we consider the family $\mathcal{QDB}_1$ of all functions which are in the intersection of the classes $\mathcal{Q}$, $\mathcal{D}$ and $\mathcal{B}_1$. Because $\mathcal{DB}_1 = \mathcal{YB}_1$ it is true that $f \in \mathcal{QDB}_1$ iff f is a *Baire 1* function and $f|\mathcal{C}(f)$ is bilaterally dense in f (by $\mathcal{C}(f)$ we denote the set of points of continuity of f). In [9] T. Natkaniec proved that the following inclusion holds $\mathcal{M} \subset \mathcal{Q}$.

Example 1. Let $I = [0,1]$, $C \subset I$ be a Cantor set and for each $n \in \mathcal{N}$ let $\mathcal{J}_n$ be the family of all components of the set $I \setminus C$ of the n-th order (i.e. such components of I $\setminus C$ which length is equal to 3^{-n}). Let A $= I \setminus \bigcup\{\overline{J} : J \in \bigcup_{n=1}^{\infty} \mathcal{J}_n\}$ and $c_J = \frac{\inf J + \sup J}{2}$ where $J \in \bigcup_{n=1}^{\infty} \mathcal{J}_n$. Let f vanish on A, be equal to 1 on each of c_J, be equal to 1 on each of the points $\max\overline{J}, \min\overline{J}$ and which is linear on both intervals $[c_J, \max\overline{J}]$ and $[\min\overline{J}, c_J]$. Then f is quasi-continuous, Baire class 1 with Darboux property. But since the set A is not an F_σ set, thus $f^{-1}(\mathbb{R} \setminus \{0\})$ is not a G_δ set and consequently f is not *Baire .5*.

2. Operations.

Let $\mathcal{L}$ be a fixed class of real functions. The *maximal additive (multiplicative)* class for $\mathcal{L}$ we define as the class off all functions $f \in \mathcal{L}$ for which $f + g \in \mathcal{L}$ ($fg \in \mathcal{L}$) whenever $g \in \mathcal{L}$. The adequate classes we denote by $\mathcal{M}_a(\mathcal{L})$ and $\mathcal{M}_m(\mathcal{L})$, respectively. Moreover, let

$$\mathcal{M}_{min}(\mathcal{L}) = \{f \in \mathcal{L} : \text{ if } g \in \mathcal{L} \text{ then } \min(f,g) \in \mathcal{L}\},$$
$$\mathcal{M}_{max}(\mathcal{L}) = \{f \in L : \text{ if } g \in \mathcal{L} \text{ then } \max(f,g) \in \mathcal{L}\}.$$

In the present paper we shall prove that

$$\mathcal{M}_a(\mathcal{QDB}_1) = \mathcal{C}, \qquad \mathcal{M}_m(\mathcal{QDB}_1) = \mathcal{M},$$
$$\mathcal{M}_{max}(\mathcal{QDB}_1) = usc\,\mathcal{QD}, \quad \mathcal{M}_{min}(\mathcal{QDB}_1) = lsc\,\mathcal{QD}.$$

Lemma 1 $\mathcal{C} \subset \mathcal{M}_a(\mathcal{QDB}_1)$.

Proof. Z. Grande and L. Sołtysik showed in [4] that $\mathcal{M}_a(\mathcal{Q}) = \mathcal{C}$. Moreover A. M. Bruckner in [11] proved that $\mathcal{M}_a(\mathcal{DB}_1) = \mathcal{C}$. Thus, if $f \in \mathcal{C}$ then $f + g \in \mathcal{QDB}_1$ for each function $g \in \mathcal{QDB}_1$.

Q.E.D.

Lemma 2 $\mathcal{M} \subset \mathcal{M}_m(\mathcal{QDB}_1)$.

Proof. R. Fleissner [3] showed that $\mathcal{M} = \mathcal{M}_m(\mathcal{DB}_1)$ and T. Natkaniec in [8] proved that $\mathcal{M} = \mathcal{M}_m(\mathcal{QA})$. Because equality $\mathcal{QDB}_1 = \mathcal{QAB}_1$ holds, we obtain required inclusion.

Q.E.D.

Lemma 3 *usc* $\mathcal{QD} \subset \mathcal{M}_{max}(\mathcal{QDB}_1)$, *lsc* $\mathcal{QD} \subset \mathcal{M}_{min}(\mathcal{QDB}_1)$.

Proof. In [2] J. Farkova showed that $usc\mathcal{D} = \mathcal{M}_{max}(\mathcal{DB}_1)$ and $lsc\mathcal{D} = \mathcal{M}_{min}(\mathcal{DB}_1)$. T. Natkaniec showed in [9] that $usc\,\mathcal{QD} = \mathcal{M}_{max}(\mathcal{QD})$ and $lsc\,\mathcal{QD} = \mathcal{M}_{min}(\mathcal{QD})$. Thus $\max(f,g) \in \mathcal{QD} \cap \mathcal{DB}_1 = \mathcal{QDB}_1$ ($\min(f,g) \in \mathcal{QDB}_1$) for every functions $f \in usc\,\mathcal{QD}$ ($f \in lsc\mathcal{QD}$) and $g \in \mathcal{QDB}_1$.

Q.E.D.

Theorem 1 $\mathcal{C} = \mathcal{M}_a(\mathcal{QDB}_1)$.

Proof. We need only to prove the following inclusion: $\mathcal{M}_a(\mathcal{QDB}_1) \subset \mathcal{C}$. Let $f \in \mathcal{QDB}_1 \setminus \mathcal{C}$ and let $x \in \mathbb{R}$ be a point of discontinuity of f. We shall show that there exists a function $g \in \mathcal{QDB}_1$ such that $f + g$ is not a Darboux function. Without loss of generality we can assume that x is a left hand sided point of discontinuity of f. Because f has the Darboux property, the left hand sided cluster set $\mathcal{K}^-(f,x)$ is a nondegenerated closed interval. Thus there exists a point $y \in \mathcal{K}^-(f,x)$ such that $f(x) \neq y$.

Put

$$g(u) = \begin{cases} -f(u) & \text{if } u < x, \\ -y & \text{if } u \geq x. \end{cases}$$

Then g is in the first class of Baire, is quasi-continuous and has the Darboux property, but the function

$$f + g = \begin{cases} 0 & \text{for } u < x, \\ f(x) - y & \text{for } u \geq x, \end{cases}$$

has not the Darboux property.

Q.E.D.

Theorem 2 $\mathcal{M} = \mathcal{M}_m(\mathcal{QDB}_1)$.

Proof. By Lemma 2 we need only to prove that $\mathcal{M}_m(\mathcal{QDB}_1) \subset \mathcal{M}$, i.e. that if $f \in \mathcal{QDB}_1 \setminus \mathcal{M}$ then there exists a $g \in \mathcal{QDB}_1$, such that $fg \notin \mathcal{QDB}_1$. If $f \notin \mathcal{M}$ then there exists a point x of discontinuity of f, such that either

(i) for some unilateral neighbourhood U of x, $f(u) \neq 0$ for $u \in U$

or

(ii) $f(x) \neq 0$.

Consider the first case. We can assume that f is discontinuous from the left at x and $f(u) \neq 0$ for any point $u \in (x-\delta, x)$, where $\delta > 0$. Choose $y \neq f(x)$, $y \in \mathcal{K}^-(f,x)$.
Define g by

$$g(u) = \begin{cases} (f(x-\delta))^{-1} & \text{if } u \leq x-\delta, \\ (f(u))^{-1} & \text{if } u \in (x-\delta, x), \\ y^{-1} & \text{if } u \geq x. \end{cases}$$

It is easy to verify that $g \in \mathcal{QDB}_1$. But

$$fg(u) = \begin{cases} f(u) \cdot (f(x-\delta))^{-1} & \text{if } u \leq x-\delta, \\ 1 & \text{if } u \in (x-\delta, x), \\ f(u) \cdot y^{-1} & \text{if } u \geq x. \end{cases}$$

Since $fg(u) = 1$ for each $u \in (x-\delta, x)$ and $fg(x) \neq 1$, fg has not the Darboux property.

Now suppose that f is discontinuous from the left at x and $a = f(x) \neq 0$. Put

$$g(u) = \begin{cases} 2a - f(u) & \text{if } u < x, \\ 2a & \text{if } u \geq x. \end{cases}$$

Evidently, g is quasi-continuous, *Baire 1* and has the Young property. Hence $g \in \mathcal{QDB}_1$. But observe that the function

$$fg(u) = \begin{cases} (2a - f(u))f(u) & \text{if } u < x, \\ 2af(u) & \text{if } u \geq x, \end{cases}$$

has not the Darboux property. Indeed, we have

$$\begin{aligned} (fg)(u) &= (2a - f(u))f(u) = 2af(u) - f^2(u) \\ &= 2a(a + f(u) - a) - (a + f(u) - a)^2 \\ &= 2a(a + f(u) - a) - (a^2 + 2a(f(u) - a)) - (f(u) - a)^2 \\ &= 2a^2 + 2a(f(u) - a) - a^2 - 2a(f(u) - a) - (f(u) - a)^2 \\ &= 2a^2 - a^2 - (f(u) - a)^2 \\ &= a^2 - (f(u) - a)^2 \text{ for each } u < x. \end{aligned}$$

This shows that $(fg)(u) \leq a^2 = f(x)^2 < fg(x)$ and consequently (fg) cannot have the Darboux property. Finally we obtain the required equality

$$\mathcal{M}_m(\mathcal{QDB}_1) = usc\,\mathcal{QD}.$$

Q.E.D.

Theorem 3 $\mathcal{M}_{max}(\mathcal{QDB}_1) = usc\,\mathcal{QD}$, $\mathcal{M}_{min}(\mathcal{QDB}_1) = lsc\,\mathcal{QD}$.

Proof. We shall prove that the following inclusion holds:

$$\mathcal{M}_{max}(\mathcal{QDB}_1) \subset usc\,\mathcal{QD}.$$

The opposite inclusion follows by Lemma 3.
Let $f \in \mathcal{QDB}_1$ be arbirtrary and let x be such that $f(x) < \limsup_{y\to x} f(y)$.
Without loss of generality we can assume that f is discontinuous from the left at x. Put $y = \frac{1}{2}\cdot(\max \mathcal{K}^-(f,x) + f(x))$ if $\max \mathcal{K}^-(f,x) \neq \infty$ and $y = f(x) + 1$ otherwise. Define g by

$$g(u) = \begin{cases} 2y - f(u) & \text{if } u < x, \\ f(x) & \text{if } u \geq x. \end{cases}$$

Obviously g is quasi-continuous, *Baire 1* and satisfies the Young condition at any point $u \neq x$. Since $f \in \mathcal{DB}_1$ and $y \in \mathcal{K}^-(f,x)$, so if $\max \mathcal{K}^-(f,x) \neq \infty$ (if $\max \mathcal{K}^-(f,x) = \infty$) then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \nearrow x$ and $\lim_{n\to\infty} f(x_n) = \max \mathcal{K}^-(f,x)$ ($\lim_{n\to\infty} f(x_n) = f(x) + 2$). Then

$$\begin{aligned} &\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}(2y - f(x_n) = \\ &= \lim_{n\to\infty}(\max \mathcal{K}^-(f,x) + f(x) - f(x_n)) = \\ &= \max \mathcal{K}^-(f,x) + f(x) - \lim_{n\to\infty} f(x_n) = f(x) \end{aligned}$$

$$\left(\begin{aligned} \lim_{n\to\infty} g(x_n) &= 2(f(x)+1) - \lim_{n\to\infty} f(x_n) \\ &= 2f(x) + 2 - f(x) - 2 \\ &= f(x) \end{aligned} \right)$$

Moreover since g is constant from the right at x we obtain that g has the Young property at x.
Since $\max(f,g) \geq y > f(x)$ for $u < x$, so $\max(f,g)$ has not the Darboux property. This proves that $\mathcal{M}_{max}(\mathcal{QDB}_1) = usc\,\mathcal{QD}$.
The inclusion $lsc\,\mathcal{QD} \subset \mathcal{M}_{min}(\mathcal{QDB}_1)$ is proved in Lemma 3. The opposite inclusion $\mathcal{M}_{min}(\mathcal{QDB}_1) \subset lsc\,\mathcal{QD}$ follows immediately from the facts $\mathcal{M}_{max}(\mathcal{QDB}_1) \subset usc\,\mathcal{QD}$, $\min(f,g) = -\max(-f,-g)$ and $f \in usc$ iff $-f \in lsc$.

Q.E.D.

3. Remarks

Now we would like to call attention to some properties of considered classes of functions. Exploring a maximal multiplicative family for $\mathcal{DB}_1$ R. Fleissner defined the family $\mathcal{M}$. However, it turned out that this family is more universal, i.e. exploring other cases we get the same family $\mathcal{M}$. Exploring maximal multiplicative family for quasi-continuous functions Z. Grande in [5] defined a subfamily $\mathcal{M}_0$ of $\mathcal{M}$. T. Natkaniec in [9] proved that if $f \in \mathcal{M}$ then for every left (right) hand sided point x of discontinuity of f there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of one side handed points of continuity of f, such that $x_n \searrow x_0 (x_n \nearrow x_0)$ and $f(x_n) = 0$ for every $n \in \mathcal{N}$. On account of the above and our result that $\mathcal{M}_m(\mathcal{QDB}_1) = \mathcal{M}$ there arises a question whether $\mathcal{M}_0$ is a proper subfamily of $\mathcal{M}$ or not. The answer to this question is negative.

Remark 1 *There exists a function $f \in \mathcal{M}$ which vanishes only at points of discontinuity of f.*

Let $I = [0,1]$, $C \subset I$, $\mathcal{J}_n$ and c_J be as in the Example 1.1. Then we define

$$f(x) = \begin{cases} 1 & \text{if} \quad x = c_J, \qquad J \in \bigcup_{n=1}^{\infty} \mathcal{J}_n \\ 0 & \text{if} \quad x \in \mathrm{C} \\ \text{linear} & \text{on} \quad \text{the intervals} \quad [\min \overline{J}, c_J], [c_J, \max \overline{J}]. \end{cases}$$

It is easy to see that $f \in \mathcal{M}$, but since $f(x) \neq 0$ for any continuity point x of f, $f \notin \mathcal{M}_0$.

An easy consequence of our result is the following

Remark 2 *For every function $f \in \mathcal{QDB}_1$ the following conditions are equivalent*

(1) $f \in \mathcal{C}$,

(2) *the function* $F = (f,g) : \mathbb{R} \longrightarrow \mathbb{R}^2 (\mathrm{F(x)} = (\mathrm{f(x)}, \mathrm{g(x)}))$ *is quasi-continuous, of the first class of Baire with Darboux property for every function* $g \in \mathcal{QDB}_1$.

Proof. The implication (2)⇒(1) follows from the fact that

$$\mathcal{M}_a(\mathcal{QDB}_1) = \mathcal{C}.$$

Indeed if f is not continuous, then there exists $g \in \mathcal{QDB}_1$ such that $(f+g)$ is not Darboux. Suppose that $F = (f,g) \in \mathcal{QDB}_1$. Since the sum $(x,y) \mapsto x+y$ is a continuous function and the composition of a function from the class $\mathcal{QDB}_1$ with continuous function belongs (obviously) to $\mathcal{QDB}_1$, $f+g \in \mathcal{QDB}_1$, too, in contradiction with the choice of g.

Now we shall verify that $(f,g) \in \mathcal{QDB}_1$ for every function $f \in \mathcal{C}$ and $g \in \mathcal{QDB}_1$. Let $f \in \mathcal{C}$ and $g \in \mathcal{QDB}_1$. It is easy to see that $(f,g) \in \mathcal{QB}_1$. We shall prove that $(f,g) \in \mathcal{D}$. It is well-known (and easy to prove) that $\mathcal{C}onn \subset \mathcal{D}$. Thus we shall show that $(f,g) \in \mathcal{C}onn$. Since $g \in \mathcal{DB}_1$, Brown's theorem implies $g \in \mathcal{C}onn$. Let A be arbitrary connected subset of $\mathbb{R}$. Notice that the function $H : g|A \longrightarrow \mathbb{R}^3$ defined by $H(x,g(x)) = (x, f(x), g(x))$ for $x \in A$ is continuous. Since $g \in \mathcal{C}onn$, $g|A$ is connected and therefore $H(g|A) = (f,g)|A$ is connected, too. Thus $(f,g) \in \mathcal{C}onn$ and consequently $(f,g) \in \mathcal{QDB}_1$.

Q.E.D.

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