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# SOME PROPERTIES OF H-ALMOST CONTINUOUS MULTIFUNCTIONS

ABSTRACT. H-almost continuous multifunctions were defined in [13] and [20] as a generalization of the univocal H-almost continuous applications, defined by Husain in [5]. Some properties of the H-almost continuous multifunctions are studied in [13] - [16] and [20]. The purpose of the present paper is to investigate some properties of these multifunctions and to obtain new characterizations using the notions of preopen sets and preclosed sets generalizing the results from [1], [2], [6], [8], [9], [14], [17] - [19].

1. Introduction. The H-almost continuous multifunctions we defined in [13] as a generalizations of the univocal H-almost continuous applications defines by Husain in [5]. Smithson also defines the H-almost continuous multifunctions in [20].

The purpose of the present paper is to investigate some properties of these multifunctions and to obtain new characterizations using the notions of preopen sets and preclosed sets generalizing the results from [1], [2], [6], [8], [9], [14], [17] - [19].

The concepts of preopen sets, preclosed sets and precontinuous functions were introduced and studied in [4], [6], [8] - [11].

DEFINITION 1.1 / [13] , [20] /

Let X and Y be two topological spaces. For a multifunction $F:X \to Y$ we shall note by $F^+(G)$ and $F^-(G)$ the upper and lower inverse of the set $G \subset Y$ as in [3] , and then we have

$$F^+(G) = \{x \in X: F(x) \subset G\}; \quad F^-(G) = \{x \in X: F(x) \cap G \neq \emptyset\}$$

/a/ The multifunction $F:X \to Y$ is H-upper almost continuous (H.U.a.c) at $x \in X$ if for each open set $G \subset Y$ with $F(x) \subset G$, it follows that $x \in \text{Int Cl}[F^+(G)]$ .

/b/ The multifunction $F:X \to Y$ is H-lower almost continuous (H.l.a.c.) in $x \in X$ if for each open set $G \subset Y$ with $F(x) \cap G \neq 0$, it follows that $x \in \text{Int Cl}[F^-(G)]$.

/c/ The multifunction $F:X \to Y$ is H-almost continuous (H.a.c.) in the point $x \in X$ if it is H.u.a.c. and H.l.a.c. in x.

/d/ The multifunction $F:X \to Y$ is H.a.c. (H.u.a.c.; H.l.a.c.) if it has this property in any point $x \in X$.

DEFINITION 1.2 ([8])

Let X be a topological space and S be a subset of X.S is said to be preopen if $S \subset \text{Int}(\text{Cl } S)$ .

The complement of a preopen set is called preclosed.

The family of all preopen sets in X will be denoted by PO (X).

Every open set is preopen set.

DEFINITION 1.3 ([4])

The intersection of all preclosed sets containing a set A is called the preclosure of A and is denoted by Pcl A.

DEFINITION 1.4 ([10])

The union of all preopen sets which are contained in A is called the preinterior of A and is denoted by P-Int A.

DEFINITION 1.5

Let x be a point of a topological space X. $U \subset X$ is called a preneighbourhood of x in X if there exists $A \in PO(X)$ such that $x \in A \subset U$.

LEMMA. A set A in a toplogical space X is preclosed if and only if $Cl(Int\ A) \subset A$.

Proof. Suppose first that A is preclosed, then X-A is preopen. Therefore

$$X-A \subset Int\ (Cl\ (X-A)) = Int\ (X-Int\ A) = X-Cl\ (Int\ A)$$

and so $Cl\ (Int\ A) \subset A$.

Next suppose that $A \supset Cl\ (Int\ A)$, then

$$X-A \subset X-Cl\ (Int\ A) = Int\ (X-Int\ A) = Int\ Cl\ (X-A).$$

Hence X-A is preopen and A is preclosed.

DEFINITION 1.6 ([8])

Let X and Y be two topological spaces. The function $f:X \to Y$ is precontinuous if the inverse image of each open set in Y is preopen in X.

By analogy with the precontinuous functions we shall define the notions of precontinuous multifunctions.

DEFINITION 1.7

Let X and Y be two topological spaces.

/a/ The multifunction $F:X \to Y$ is upper precontinuous (u.p.c.) if $F^+(G) \in PO(X)$ for each open set $G \subset Y$.

/b/ The multifunction $F:X \to Y$ is lower precontinuous (l.p.c.) if $F^-(G) \in PO(X)$ for each open set $G \subset Y$.

/c/ The multifunction $F:X \to Y$ is precontinuous p.c. if it is upper and lower precontinuous.

DEFINITION 1.8

Let A be a set of a topological space X. U is a neighbourhood which intersects A if there exists an open set $V \subset X$ such that $V \subset U$ and $V \cap A \neq \emptyset$.

## 2. Characterizations

The following theorems are demonstrated in [14].

THEOREM 2.1 ([14])

For a multifunction $F:X \to Y$ the following are equivalent:

1. F is H.u.a.c.
2. For every open set $G \subset Y$ there is the relation

$$F^+(G) \subset \text{Int } \text{Cl}\left[F^+(G)\right].$$

3. For every closed set $V \subset Y$ there is the relation

$$F^-(V) \subset \text{Cl } \text{Int}\left[F^-(V)\right].$$

THEOREM 2.2 ([14])

For a multifunction $F:X \to Y$ the following are equivalent:

1. F is H.l.a.c.
2. For every open set $G \subset Y$ there is the relation

$$F^-(G) \subset \text{Int } \text{Cl}\left[F^-(G)\right].$$

3. For every closed set $V \subset Y$ there is the relation

$$F^-(V) \subset \text{Cl } \text{Int}\left[F^+(V)\right].$$

THEOREM 2.3.

For a multifunction $F:X \twoheadrightarrow Y$ the following are equivalent:

1. F is H.u.a.c.
2. F is u.p.c.
3. For every closed set $V \subset Y, F^-(V)$ is preclosed in X.
4. For each point x in X and each open set $G \subset Y$ with $F(x) \subset G$ there is a preopen set $U \subset X$ such that $x \in U$ and $F(U) \subset G$.
5. For each point x in X and for each neighbourhood V of $F(x)$, $F^+(V)$ is a pre-neighbourhood of x.
6. For each point x in X and for each neighbourhood V of $F(x)$ there is a pre-neighbourhood U of x such that $F(U) \subset V$.
7. For each subset B of $Y, F^+(\text{Int } B) \subset \text{P-Int}[F^+(B)]$.
8. For each subset B of Y, $\text{Cl Int}[F^-(B)] \subset F^-(\text{Cl } B)$.

Proof. (1) $\Leftrightarrow$ (2). Follows by Theorem 2.1 (2) and Definition 2.1.

(1) $\Leftrightarrow$ (3). Follows by Lemma and Theorem 2.1 (3).

(2) $\Rightarrow$ (4). Let $G \subset Y$ be an open set and $F(x) \subset G$. Then $x \in F^+(G)$ which is preopen by hypothesis. Put $U = F^+(G)$, then $x \in U$ and $F(U) \subset G$.

(4) $\Rightarrow$ (2). Let G be an open set in Y and $x \in F^+(G)$, then $F(x) \subset G$. Therefore, by hypothesis there exists a preopen set $U_x$ in X such that $x \in U_x$ and $F(U_x) \subset G$. Consequently, $F^+(G)$ is an union of preopen sets in X and hence is a preopen set by [8].

(4) $\Rightarrow$ (5). Let $x \in X$ and $V \subset Y$ be a neighbourhood of $F(x)$, then there is an open set $G \subset Y$ such that $G \subset V$ and $F(x) \subset G$. Then there is a preopen set $U \subset X$ containing x such

that $F(U) \subset G$, which implies $U \subset F^+(G)$. $G \subset V$ implies that $x \in U \subset F^+(G) \subset F^+(V)$ and thus $F^+(V)$ is a pre-neighbourhood of x.

(5) $\Rightarrow$ (6). Let $x \in X$ and $V \subset Y$ be a neighbourhood of $F(x)$. According to the hypothesis, $U = F^+(V)$ is a pre-neighbourhood of x and $F(U) \subset V$.

(6) $\Rightarrow$ (4). Let $x \in X$ and $G \subset Y$ be an open set such that $F(x) \subset G$. G being open set is a neighbourhood of $F(x)$ and according to the hypothesis, there is a pre- neighbourhood $U_1$ of x such that $F(U_1) \subset G$. Then there is a preopen set $U \subset U_1$ containing x such that $F(U) \subset G$.

(2) $\Rightarrow$ (7). Let B be any subset of Y. Then Int B is an open set in Y and $F^+(\text{Int } B)$ is a preopen set in X. Since $F^+(\text{Int } B) \subset$ $\subset F^+(B)$, then $F^+(\text{Int } B) \subset \text{P-Int}\left[F^+(B)\right]$.

(7) $\Rightarrow$ (2). Let B be any open set of Y, then Int B=B and $F^+(B) \subset \text{P-Int}\left[F^+(B)\right]$. Thus $F^+(B) = \text{P-Int}\left[F^+(B)\right]$ and $F^+(B)$ is a preopen set in X.

(3) $\Rightarrow$ (8). Let B be any subset of Y. Then according to the hypothesis $F^-(\text{Cl } B)$ is a preclosed set in X. $\text{Cl}B \supset B$ implies $F^-(\text{Cl } B) \supset F^-(B)$. Then by Lemma

$$\text{Cl Int}\left[F^-(B)\right] \subset \text{Cl Int}\left[F^-(\text{Cl } B)\right] \subset F^-(\text{Cl } B) .$$

(8) $\Rightarrow$ (3). Let B be any closed set of Y. Then according to the hypothesis $\text{Cl Int}\left[F^-(B)\right] \subset F^-(\text{Cl } B) = F^-(B)$ which implies by Lemma that $F^-(B)$ is a preclosed set in X.

THEOREM 2.4. For a multifunction $F: X \to Y$ the following are equivalent:

1. F is H.l.a.c.

2. $F$ is l.p.c.

3. For every closed set $V \subset Y, F^{+}(V)$ is preclosed in $X$.

4. For each point $x$ in $X$ and for each open set $G \subset Y$ with $F(x) \cap G \neq \emptyset$ there is a preopen set $U \subset X$ such that $x \in U$ and $F(y) \cap G \neq \emptyset$, $\forall\ y \in U$.

5. For each point $x$ in $X$ and for each neighbourhood $V$ which intersects $F(x)$, there is a pre-neighbourhood $U$ of $x$ such that $F(y) \cap V \neq \emptyset$, $\forall\ y \in U$.

6. For each point $x$ in $X$ and for each neighbourhood $V$ which intersects $F(x)$, $F^{-}(V)$ is a pre-neighbourhood of $x$.

7. For each subset $B$ of $Y$, $F^{-}(\text{Int } B) \subset \text{P-Int}\,[F^{-}(B)]$.

8. For each subset $B$ of $Y$, $\text{Cl Int}\,[F^{+}(B)] \subset F^{+}(\text{Cl } B)$.

9. For each subset $A$ of $X$, $F(\text{Pcl } A) \subset \text{Cl } F(A)$.

10. For each subset $B$ of $Y$, $\text{Pcl}\,[F^{+}(B)] \subset F^{+}(\text{Cl } B)$.

11. For each subset $A$ of $X$, $F(\text{Cl Int } A) \subset \text{Cl } F(A)$.

12. For each open set $A$ of $X$, $F(\text{Cl } A) \subset \text{Cl } F(A)$.

<u>Proof.</u> (1) $\Leftrightarrow$ (2) . Follows by Theorem 2.2 (2) and Definition 1.2.

(1) $\Leftrightarrow$ (3) . Follows by Lemma and Theorem 2.2 (3) .

(2) $\Rightarrow$ (4) . Let $G \subset Y$ be an open set and $F(x) \cap G \neq \emptyset$ Then $x \in F^{-}(G)$ which is preopen by hypothesis. Put $U = F^{-}(G)$, then $x \in U$ and $F(y) \cap G \neq \emptyset$, $\forall\ y \in G$.

(4) $\Rightarrow$ (2) . Let $G$ be an open set in $Y$ and $x \in F^{-}(G)$, then $F(x) \cap G \neq \emptyset$. Therefore, by hypothesis there exists a preopen set $U_x$ in $X$ such that $x \in U_x$ and $F(y) \cap G \neq \emptyset$, $\forall\ y \in U_x$. Then $x \in U_x \subset F^{-}(G)$.
Consequently, $F^{-}(G)$ is an union of preopen sets in $X$ and hence is a preopen set by [8].

(4) $\Rightarrow$ (5) . Let $x \in X$ and $V \subset Y$ be a neighbourhood which intersets $F(x)$, then there is an preopen set $G \subset Y$ such

that $G \subset V$ and $F(x) \cap G \neq \emptyset$. Then there is a preopen set $U \subset X$ containing $x$ such that $F(y) \cap G \neq \emptyset$, $\forall y \in U$, which implies $U \subset F^-(G)$. $G \subset V$ implies that $x \in U \subset F^-(G) \subset F^-(V)$ and thus $F^-(V)$ is a pre- neighbourhood of $x$.

(5) $\Rightarrow$ (6). Let $x \in X$ and $V \subset Y$ be a neighbourhood which intersects $F(x)$. According to the hypothesis, $U = F^-(V)$ is a pre-neighbourhood of $x$ and $F(y) \cap V \neq \emptyset$, $\forall y \in U$.

(6) $\Rightarrow$ (4). Let $x \in X$ and $G \subset Y$ be an open set such that $F(x) \cap G \neq \emptyset$. $G$ being open then $G$ is a neighbourhood which intersects $F(x)$ and according to the hypothesis, there is a pre-neighbourhood $U_1$ of $x$ such that $F(y) \cap G \neq \emptyset$, $\forall y \in U_1$. Then there is a preopen set $U \subset U_1$ containing $x$ such that $F(z) \cap G \neq \emptyset$, $\forall z \in U$.

(2) $\Rightarrow$ (7). Let $B$ be any subset of $Y$, then Int $B$ is an open set and $F^-(\text{Int } B)$ is a preopen set in $X$. Since $F^-(\text{Int } B) \subset F^-(B)$, then $F^-(\text{Int } B) \subset \text{P-Int}\,[F^-(B)]$.

(7) $\Rightarrow$ (2). Let $B$ be any open subset of $Y$, then Int $B = B$ and $F^-(B) \subset \text{P-Int}\,[F^-(B)]$. Thus $F^-(B) = \text{P-Int}\,[F^-(B)]$ and $F^-(B)$ is a preopen set.

(3) $\Rightarrow$ (8). Let $B$ be any subset of $Y$. According to the hypothesis $F^+(\text{Cl } B)$ is a preclosed set in $X$. Then by Lemma

$$\text{Cl Int}\,[F^+(B)] \subset \text{Cl Int}\,[F^+(\text{Cl } B)] \subset F^+(\text{Cl } B).$$

(8) $\Rightarrow$ (3). Let $B$ be any closed subset of $Y$. According to the hypothesis Cl Int $[F^+(B)] \subset F^+(\text{Cl } B) = F^+(B)$ which implies by Lemma that $F^+(B)$ is a preclosed set in $X$.

(3) $\Rightarrow$ (9). Let $A$ be any subset of $X$. Then by $A \subset F^+(F(A))$ follows that $A \subset F^+(\text{Cl } F(A))$. According to the hypothesis $F^+(\text{Cl } F(A))$ is a preclosed set in $X$. Then Pcl $A \subset F^+(\text{Cl } F(A))$.

Consequently

$$F(\mathrm{Pcl}\ A) \subset F(F^{+}(\mathrm{Cl}\ F(A))) \subset \mathrm{Cl}\ F(A).$$

(9) $\Rightarrow$ (3). Let B be any closed subset of Y. According to the hypothesis

$$F(\mathrm{Pcl}\ F^{+}(B)) \subset \mathrm{Cl}\ F(F^{+}(B)) \subset \mathrm{Cl}\ B = B.$$

Then $\mathrm{Pcl}\ [F^{+}(B)] \subset F^{+}(B)$ and by Lemma 2.3 (1) of [4], $F^{+}(B)$ is preclosed.

(9) $\Rightarrow$ (10). Let B be any subset of Y. Replacing A by $F^{+}(B)$ we get by hypothesis

$$F(\mathrm{Pcl}\ [F^{+}(B)]) \subset \mathrm{Cl}\ F(F^{+}(B)) \subset \mathrm{Cl}\ B$$

hence $\mathrm{Pcl}\ [F^{+}(B)] \subset F^{+}(\mathrm{Cl}\ B)$.

(10) $\Rightarrow$ (9). Let $B = F(A)$ be, where A is a subset of X. Then according to the Lemma 2.3(2) of [4], we have

$$\mathrm{Pcl}\ A \subset \mathrm{Pcl}\ F^{+}(B) \subset F^{+}(\mathrm{Cl}\ B) \subset F^{+}[\mathrm{Cl}\ F(A)]$$

hence $F(\mathrm{Pcl}\ A) \subset \mathrm{Cl}\ F(A)$.

(8) $\Rightarrow$ (11). Let A be any subset of X. Replacing B by $F(A)$ we get by hypothesis

$$\mathrm{Cl}\ \mathrm{Int}\ A \subset \mathrm{Cl}\ \mathrm{Int}\ [F^{+}(B)] \subset F^{+}(\mathrm{Cl}\ B)$$

hence

$$F(\mathrm{Cl}\ \mathrm{Int}\ A) \subset F(F^{+}(\mathrm{Cl}\ B)) \subset \mathrm{Cl}\ B = \mathrm{Cl}\ F(A).$$

(11) $\Rightarrow$ (3). Let B be a closed set in Y. Let $A = F^{+}(B)$ be. According to the hypothesis we get

$$F[\mathrm{Cl}\ \mathrm{Int}\ A] \subset \mathrm{Cl}\ F(A) \subset \mathrm{Cl}\ F(F^{+}(B)) \subset \mathrm{Cl}\ B = B$$

hence

$$\mathrm{Cl}\ \mathrm{Int}\ A \subset F^{+}(F(\mathrm{Cl}\ \mathrm{Int}\ A)) \subset F^{+}(B)$$

and $\mathrm{Cl}\ \mathrm{Int}\ [F^{+}(B)] \subset F^{+}(B)$. According to the Lemma $F^{+}(B)$ is a preclosed set in X.

(9) $\Rightarrow$ (12). Let A be an open set in X. Then by Theorem 2.4 of [4] Pcl A = Cl A, hence F (Cl A) $\subset$ Cl F (A).

(12) $\Rightarrow$ (11). Let A be any subset of X. Then Int A is a open set of X. According to the hypothesis

$$F(\text{Cl Int } A) \subset \text{Cl } F(\text{Int } A) \subset \text{Cl } F(A).$$

COROLLARY 2.1. Let X and Y be two topological spaces. For a univocal application f:X$\rightarrow$Y the folowing are equivalent:

1. f is a.c.H.
2. f is precontinuous. (Theorem 1 (i) ; [8]).
3. For every closed set V $\subset$ Y, $f^{-1}$ (V) is preclosed in X . (Theorem 1 (iv) ; [8]) .
4. For every point x in X and each open set G $\subset$ Y with f (x) $\in$ G, there is a preopen set U $\subset$ X such that x $\in$ U and f (U) $\subset$ G. ( Theorem 1 (ii) ; [8]) .
5. For each point x in X, the inverse of every neighbourhood of f (x) is a pre- neighbourhood of x. (Theorem 1 (3); [18]).
6. For each point x in X and each neighbourhood V of f (x) there is a pre-neighbourhood U of x such that f (U) $\subset$ V. (Theorem 1 (4) ; [18]).
7. For each subset B of Y, $f^{-1}$ (Int B) $\subset$ P-Int [$f^{-1}$ ( B)] . (Theorem 1 (8) ; [18]).
8. For each subset B of Y, Cl Int [$f^{-1}$ (B)] $\subset$ $f^{-1}$ (Cl B) . (Theorem 1 (v) ; [8]).
9. For each subset A of X, f (Pcl A) $\subset$ Cl f( A). (Theorem 1(5) ; [18]).

10. For each subset B of Y, Pcl $f^{-1}(B) \subset f^{-1}(\mathrm{Cl}\ B)$.
(Theorem 1 (6), [18]).
11. For each subset B of Y, f (Cl Int A) $\subset$ Cl f(A).
(Theorem 1 (vi) ; [8]).
12. For each open set A of X, f (Cl A) $\subset$ Clf (A).
(Theorem 6 (3) ; [19])

## 3. Some properties of H.a.c. multifunctions

DEFINITION 3.1. Let X and Y be two topological spaces.
If $F : X \to Y$ is a multifunction, we shall understand by Pcl F that multifunction that takes x into Pcl F (x).
For the multifunctions H.l.a.c. we shall extend a proposition demonstrated in [1] (Theorem $I_1$) for the multifunction l.s.c.

THEOREM 3.1. If the multifunction $F: X \to Y$ is H.l.a.c. then the multifunction Pcl $F : X \to Y$ is H.l.a.c. as weel.

Proof. Let G be an open set from Y, then

$(\mathrm{Pcl}\ F)^-(G) = \{x \in X : \mathrm{Pcl}\ F(x) \cap G \neq \emptyset\}$.

Let $x_0 \in (\mathrm{Pcl}\ F)^-(G)$, so Pcl $F(x_0) \cap G \neq \emptyset$. There is then $y \in \mathrm{Pcl}\ F(x_0) \cap G$, so $y \in \mathrm{Pcl}\ F(x_0)$ and $y \in G$. From $y \in \mathrm{Pcl}\ F(x_0)$ according to the Lemma 2.2 of [4] there follows that whichever the preopen set V of Y containing y were, $V \cap F(x_0) \neq \emptyset$.
As G is open in Y, then G is preopen, hence $F(x_0) \cap G \neq \emptyset$ which implies that $x_0 \in F^-(G)$. Then $(\mathrm{Pcl}\ F)^-(G) = F^-(G)$. As F is H.l.a.c. then $F^-(G) \in PO(X)$ and according to the theorem 2.4, implication (2) $\Rightarrow$ (1), Pcl F is H.l.a.c.

DEFINITION 3.2 ([7]). A subset S is said to be semi-open if there exists an open set U of X such that $U \subset S \subset \mathrm{Cl}\ U$.

THEOREM 3.2. The multifunction $F:X \to Y$ is H.l.a.c. if and only if $F(\mathrm{Cl}\ A) \subset \mathrm{Cl}\ F(A)$ for each semi-open set $A \subset X$.

Proof. If A is semi-open then by Theorem 2.4 of [4], Pcl A=Cl A. If F is H.l.a.c. then according to the Theorem 2.4, implication (1) $\Rightarrow$ (9), $F(\mathrm{Cl}\ A) = F(\mathrm{Pcl}\ A) \subset \mathrm{Cl}\ F(A)$.

Let A be a set in X.Then Int A is an open set, thus Int A is semi-open in X and

$F(\mathrm{Cl}\ (\mathrm{Int}\ A)) \subset \mathrm{Cl}\ F(\mathrm{Int}\ A) \subset \mathrm{Cl}\ F(A)$. By Theorem 2.4, implication (9) $\Rightarrow$ (1), F is H.l.a.c.

COROLLARY 3.1. The univocal application $f:X \to Y$ is H.a.c if and only if $f(\mathrm{Cl}\ A) \subset \mathrm{Cl}\ f(A)$ for each semi-open set $A \subset X$. (Proposition 3.1 (c); [6]).

THEOREM 3.3. If $F:X \to Y$ is a multifunction so that:

1. F is point - compact,
2. F is H.u.a.c.,
3. Y is Hausdorff space,

then the graph $G(F)$ is preclosed in $X \times Y$.

Proof. Let $(x,y) \in X \times Y - G(F)$. Then we have $y \notin F(x)$. Let $F(x) = \bigcup_{i \in I} \{y_i\}$ be. Since Y is Hausdorff for each pair $(y_i, y)$ there exist disjoint open sets $U_i$ and $V_i$ such that $y_i \in U_i$ and $y \in V_i$.

F being punctually compact and $\{U_i : i \in I\}$ being an open covering of F(x) there is a finite sub-covering $\{U_j : j = 1,2,\ldots,n\}$ of $F(x)$. Let $U = \bigcup_{j=1}^{n} U_j$ and $V = \bigcap_{j=1}^{n} V_j$. Then U and V are open sets, $F(x) \subset U$, $y \in V$ and $U \cap V = \emptyset$. The multifunction F being H.u.a.c. by Theorem 2.3, implication (1) $\Rightarrow$ (4), there is $D \in PO(X)$ such that $x \in D$ and $F(D) \subset U$, thus $V \cap F(D) = \emptyset$ and $(x,y) \in D \times V \subset X \times Y - G(F)$. By Lemma 2 of [17] $(x,y) \in D \times V \in PO(X \times Y)$ and by [8] $X \times Y - G(F) \in PO(X \times Y)$. Thus $G(F)$ is a preclosed set of $X \times Y$.

COROLLARY 3.2. If $f: X \to Y$ is a H.a.c. univocal application and Y is Hausdorff, then $G(f)$ is a preclosed set in $X \times Y$.
(Theorem 2.2, [9])

DEFINITION 3.2. A space X is said to be pre-$T_2$ if for each pair of distinct points x and y in X there exist disjoint pre-open sets U and V in X such that $x \in U$ and $y \in V$.
Every $T_2$ - space is pre-$T_2$ space.

DEFINITION 3.3. ([12], [20]) Let X and Y be two topological spaces. The multifunction $F: X \to Y$ is upper weakly continuous (u.w.c.) in the point $x \in X$, if for every open set $G \subset Y$ with $F(x) \subset G$, there exists an open set $U \subset X$ containing x, so that $F(U) \subset Cl\, G$

THEOREM 3.4. If X is a topological space and for each pair of different points $x_1$ and $x_2$ from X, there is a H.u.a.c. multifunction $F: X \to Y$ with Y a $T_4$ space so that

1. F is punctually closed.
2. $F(x_1) \cap F(x_2) = \emptyset$,

then X is a pre- $T_2$ space.

<u>Proof.</u> Let $x_1 \neq x_2$ be. Then as Y is a $T_4$ space, F punctually closed and $F(x_1) \cap F(x_2) = \emptyset$, there are two open sets $V_1$ and

$V_2$ with $F(x_i) \subset V_i$, i=1,2 so that $V_1 \cap V_2 = \emptyset$. As F is H.u.a.c. in $x_1$ and $x_2$ by Theorem 2.3, implication (1)⇒(4), follows that there are two preopen sets $U_1, U_2 \subset X$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $F(U_i) \subset V_i$, i= 1,2. From $V_1 \cap V_2 = \emptyset$ follows that $F(U_1) \cap F(U_2) = \emptyset$ which implies that $U_1 \cap U_2 = \emptyset$, that is X is a pre-$T_2$ space.

THEOREM 3.5. If $F_1$ and $F_2$ are two multifunctions defined on the topological space X with values in a $T_4$ topological space Y so that:

1. $F_i$ are punctually closed, i=1,2;
2. $F_1$ is u.w.c.;
3. $F_2$ is H.u.a.c.,

then the set $\{x \in X : F_1(x) \cap F_2(x) \neq \emptyset\}$ is preclosed in X.

Proof. Let $A = \{x \in X: F_1(x) \cap F_2(x) \neq \emptyset\}$ be. We shall show that the set X-A is preopen. Let $x \in X-A$ Then we shall have $F_1(x) \cap F_2(x) = \emptyset$. As $F_i$, i = 1,2 are punctually closed and Y is a $T_4$ space, there are open sets $V_i$, i=1,2 so that $F_i(x) \subset V_i$, i = 1,2 and $V_1 \cap V_2 = \emptyset$. This implies Cl $V_1 \cap V_2 = \emptyset$. As $F_1$ is u.w.c. there is an open set $U_1 \subset X$ so that $x \in U_1$ and $F(U_1) \subset$ $\subset$ Cl $(V_1)$. As $F_2$ is H.u.a.c. by Theorem 2.3, implication (1) ⇒ (4) follows that there is a preopen set $U_2 \subset X$ containing x such that $F(U_2) \subset V_2$. Let $U = U_1 \cap U_2$, then by Lemma G of [9], $U \in PO(X)$. Moreover, $x \in U \subset X-A$ because if $y \in U$, then F (y Cl $V_1$ and $F_2(y) \subset V_2$ or Cl $V_1 \cap V_2 = \emptyset$, so $F_1(U) \cap F_2(U)$ $= \emptyset$ then $y \in X-A$. From $x \in U \subset X-A$ there follows by [8] that X-A is a preopen set, so A is a preclosed set.

THEOREM 3.6. Let $F_1 : X_1 \to Y$ and $F_2 : X_2 \to Y$ be two multifunctions with Y a $T_4$ space so that:

1. $F_i$ are point - closed, i = 1,2.
2. $F_1$ is u.w.c.
3. $F_2$ is H.u.a.c.,

then the set $\{(x_1, x_2) : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is a preclosed set in the space $X_1 \times X_2$.

Proof. Let $A = \{(x_1, x_2) : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ be. We shall show that the set $X_1 \times X_2 - A$ is preopen. Let $(x_1, x_2) \notin A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. As Y is a $T_4$ space and $F_i$, i = 1,2 are point closed, there are open sets $V_i \subset Y$, i= 1,2 with $F_i(x_i) \subset \subset V_i$, i = 1,2 and $V_1 \cap V_2 = \emptyset$. This implies $Cl\ V_1 \cap V_2 = \emptyset$. As $F_1$ is u.w.c. we shall have by Theorem 6 (2) of [12] that $x \in F_1^+(V_1) \subset Int\ F_1^+(Cl\ V_1)$. As $F_2$ is H.u.a.c. we have $x \in F^+(V_2)$, where by Theorem 2.3, implication (1) $\Rightarrow$ (2) $F_2^+(V_2) \in PO(X_2)$. Let $U = Int\ F_1^+(Cl\ V_1) \times F_2^+(V_2)$ be. Then by Lemma 2 of [17] $U \in PO(X_1 \times X_2)$. If $(y_1, y_2) \in U$, then $y_1 \in Int\ F_1^+(Cl\ V_1)$ and $y_2 \in F_2^+(V_2)$ so $F_1(y_1) \subset Cl\ V_1$ and $F_2(y_2) \subset V_2$ or $Cl\ V_1 \cap V_2 = \emptyset$. So $(x_1, x_2) \in U \subset X_1 \times X_2 - A$ and by [8] $X_1 \times X_2 - A$ is a preopen set in $X_1 \times X_2$ and so A is a preclosed set in $X_1 \times X_2$.

The following theorems are proved similarly:

THEOREM 3.7. If $F_i$, i=1,2 are two multifunctions defined on the topological space X with values in a $T_4$ topological space Y so that:

1. $F_i$, i=1,2 are punctually closed.
2. $F_i$, i=1,2 are H,u.a.c.,

then the set $\{x \in X : F_1(x) \cap F_2(x) \neq \emptyset\}$ is preclosed in X.

THEOREM 3.8. Let $F_1 : X_1 \to Y$ and $F_2 : X_2 \to Y$ be two multifunctions with Y a $T_4$ space so that:

1. $F_i$ , i=1,2 are punctually closed.
2. $F_i$ , i=1,2 are H,u.a.c.

then the set $\{(x_1, x_2) : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is a preclosed set in the space $X_1 \times X_2$.

DEFINITION 3.4([17]). A topological space $(X,T)$ is said preconnected if $X = A \cup B, A \in PO(X), B \in PO(X), A \neq \emptyset, B \neq \emptyset \Rightarrow A \cap B = \emptyset$.

Every preconnected space is connected.

THEOREM 3.9. If the multifunction $F : X \to Y$ is a surjective, punctually connected and H.u.a.c. ( H.l.a.c.) and if X is a preconnected space, the Y is connected.

Proof. It is similar to the proof of the Theorem 3.1 of [2].

COROLLARY 3.3. Let $f : X \to Y$ be an a.c.H. univocal surjection. If X is preconnected, then $Y = f(X)$ is connected.(Theorem 8 ; [27]).

REFERENCES

[1] T.Banzaru, Aplicaţii multivoce şi spaţii M-produs, Bull. şt. techn. Instit. Politehn. "T.Vuia", Timişoara,17(31), 1(1972) 17-23

[2] T.Banzaru, Aspura unor proprietăţi ale aplicaţiilor multivoce, Stud.Cerc.Matem., 24,10 (1972), 1503-1510

[3] C.Berge, Espaces topologiques. Fonctions multivoques. Dunod, Paris, 1959

[4] S.N.El Deeb, I.A.Husanein, A.S.Mashhour, T.Noiri, On P-regular spaces, Bull.Math.Soc.Sci.Mathe.R.S.R., 27 (75), 4 (1983) 311-315

[5] T.Husain, Almost continuous mappings, Prace Mat., 10 (1966), 1-7

[6] D.S.Jankovic, A note on mappings of extremally diesconnected spaces, Acta Math. Hung., 46, 1-2 (1985), 83-92

[7] N.Levine, Semi-open sets and semi-continuity in topological spaces, Amer.Math.Monthly, 70 (1963), 36-41

[8] A.S.Mashhour, M,E.Abd El-Mousef, S.N.El Deeb, On precontinuous and weak precontinuous mappings, Proc.Math.Phys., Egypt, 51 (1981)

[9] A.S.Mashhour, I.A. Hasanein, S.N.El Deeb , A note on semi-continuity and precontinuity, Indian J,pure appl. Math.13(10) (1982) , 1119-1123

[10] A.S.Mashhour, M.E. Abd El-Mousef, I.A. Hasanein, On pretopological spaces, Bull. Math.Soc.Sci. Math.R.S.R., 28 (76), 1 , (1984) , 39-45

[11] T.Noiri, Hiperconnectedness and preopen sets, Rev. Roumaine Math. pures appl., 29,4 (1984) , 329-334

[12] V.Popa, Weakly continuous multifunctions, Boll. U.M.I. (5), 15-A, (1978) , 379-388

[13] V.Popa, Asupra unor proprietati ale multifunctiilor cvasicontinue si aproape continue. Stud.Cerc.Matem, 30,4 (1978),

441-446

[14] V.Popa, Multifuncţii H-aproape continue, Stud.Cerc.Matem. 32,1 (1980) 103-109

[15] V.Popa, On some weakened forms of continuity for multifunctions, Matemat. vesnik 36 (1984), 339-350

[16] V.Popa, Sur certaines formes faibles de continuité pour les multifonctions, Rev. Roumaine Mathe pures appl., 30,7 (1985) 539-546

[17] V.Popa, Properties of H-almost continuous functions, Bull. Math. Soc.Sci.Math. R.S.R., 31 (79), 2 (1987) 163-168

[18] V.Popa, Characterizations of H-almost continuous functions, (To appear, Glasnik matemat.) .

[19] D.A.Rose, Weak continuity and almost continuity, Internaţ. J.Math. and Math.Sci. 7,2 (1984), 311-318

[20] R.E.Smithson, Almost and weak continuity for multifunctions, Bull. Calcutta Math.Soc.70 (1978), 383-390

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