ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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TWO REMARKS ON ALMOST CONTINUOUS FUNCTIONS

ABSTRACT. If $f:R \to R$ is a continuous function and $g:R \to R$ is an almost continuous function then $f+g$, $f\cdot g$, $\max(f,g)$ and $\min(f,g)$ are almost continuous.

Let us establish some of the terminology to be used. R denotes the real line and $I = \langle 0,1\rangle$. Let X,Y and Z be metric spaces. A function $f:X \to Y$ is almost continuous iff every open subset of $X \times Y$ containing f contains a continuous function with domain X. No distinction is made between a function and its graph.

A closed set $K \subset X \times Y$ is said to be a blocking set for a function $f:X \to Y$ iff $K \cap f = \emptyset$ for any continuous function $g:X \to Y$. It is easy to see that f is not almost continuous function iff there exists a blocking set $K \subset X \times Y$ for f.

PROPOSITION 1. If $f:X \to Y$ is continuous and $g:X \to Z$ is almost continuous then $h = (f,g) : X \to Y \times Z$ is almost continuous.

P r o o f. Suppose that $f:X \to Y$ is continuous, $g:X \to Z$ is almost continuous and $h : (f,g)$ is not almost continuous. Then there exists a blocking set $K \subset X \times Y \times Z$ for h. Let us put

$$F = \{(x,y) \in X \times Z \; ; \; (x, f(x), y) \in K\}.$$

First, observe that F is a closed subset of $X \times Z$. Indeed, let (x_n, y_n) be a sequence of points of F and $\lim_{n\to\infty} (x_n, y_n) = (x,y)$. Then $(x_n, f(x_n), y_n) \in K$ for each $n \in N$. Since of is continuous and $\lim_{n\to\infty} x_n = x$, we obtain $\lim_{n\to\infty} f(x_n) = f(x)$ and consequently, $\lim_{n\to\infty} (x_n, f(x_n), y_n) = (x, f(x), y)$. Hence $(x,y) \in F$.

Next, notice that $F \cap k \neq \emptyset$ for every continuous functions $k:X \to Z$. Indeed, if $k:X \to Z$ is continuous then $(f,k): X \to Y \times Z$ is continuous too. Then there exists $x \in X$ such that $(x, f(x), k(x)) \in K$ and thus $(x, k(x)) \in F$.

Now we shall verify that $F \cap g = \emptyset$. Suppose that $(x, g(x)) \in F$ for some $x \in X$. Then $(x, f(x), g(x)) \in K$, i.e. $(x, h(x)) \in K$, contradicting the fact that K is a blocking set for h. Consequently, F is a blocking set for g, contradicting almost continuity of this function.

COROLLARY. If $f:X \to R$ is a continuous function and $g:X \to R$ is an almost continuous function then the functions $f+g$, $f \cdot g$, $\max(f,g)$ and $\min(f,g)$ are almost continuous.

P r o o f. It is well-known that the superposition $g \circ f:X \to Z$ of almost continuous function $f:X \to Y$ and continuous function $g:Y \to Z$ is almost continuous [KG]. Since the fuctions sum, product, maximum and minimum are continuous, $f+g$, $f \cdot g$, $\max(f,g)$ and $\min(f,g)$ are almost continuous for every continuous function $f:X \to R$ and for every almost continuous function $g:X \to R$.

K.Kellum has been proved that each real-valued function defined on a real interval can be expresed as a sum of two almost continuous functions [K]. The same result holds with respect to products when the given function is always positive or always negative. It is showed in [N] that for every function $f:I \to R$ there exist almost continuous functions f_1, f_2, f_3, f_4 such that $f = \min(\max(f_1, f_2), \max(f_3, f_4))$. Thus the sum (product, max and min) of almost continuous functions is not almost continuous, and therefore there exist almost continuous function $f,g:I \to R$ for which the function $h = (f,g) : I \to R \times R$ is not almost continuous.

PROBLEMS. (i) Must $g:X \to R$ be continuous when $g + f$ (respectively $g \cdot f$, $\max(f,g)$) is almost continuous for every almost continuous function $f:X \to R$?

(ii) Must $g:X \to Z$ be continuous when $h = (f,g) : X \to Y \times Z$ is almost continuous for every almost continuous function $f:X \to Y$?

Prof. Grande recently has proved that every measurable (with the Baire property) function $f:I \to R$ is equal to the sum of two almost continuous, measurable (with the Baire property) functions [G]. We use of his method in the proof of the next theorem (see [N]).

PROPOSITION 2. For every measurable (with the Baire property) function $f:I \to R$ there exist almost continuous, measurable (with the Baire property) functions f_1, f_2, f_3, f_4 such that $f = \min(\max(f_1, f_2), \max(f_3, f_4))$.

P r o o f. Let $Z \subset I$ be a set with the following properties:

(i) Z has Lebesgue measure zero,

(ii) Z is of the first category,

(iii) $Z \cap (a,b)$ has the cardinality c (the continuum) for each intervals $(a,b) \subset I$.

Notice that for every blocking set $K \subset I \times R$ the projection $p_1(K)$ includes a non-empty interval $[KG]$. Therefore the set $Z \cap p_1(K)$ has the cardinality c for each blocking sets K. Let $K_0, K_1, \dots,$ $K_\alpha, \dots$ $(\alpha < c)$ be a well-ordering of all blocking sets in $I \times R$. By induction we can choose a sequence of points $(x_{\alpha,i}, y_{\alpha,i})$ $(\alpha < c, 1 \leq i \leq 4)$ such that $(x_{\alpha,i}, y_{\alpha,i}) \in K_\alpha$, $x_{\alpha,i} \in Z$ and if $x_{\alpha,i} = x_{\beta,j}$ then $\alpha = \beta$ and $i = j$. Let us define $A_i = \{ x_{\alpha,i} : \alpha < c\}$ for $i = 1,2,3,4$. Observe that $A_i \cap A_j = \emptyset$ for $i \neq j$.

Let us put

$$f_i(x) = \begin{cases} y_{\alpha,i} & \text{for } x = x_{\alpha,i}, \quad \alpha < c, \\ f(x) & \text{for } x \in A_i . \end{cases} \qquad i = 1,2,3,4.$$

Fix $i \in \{1,2,3,4\}$. Since the set $\{x: f(x) \neq f_i(x)\} \subset A_i \subset Z$, the set Z has measure zero (is of the first category) and f is measurable (has the Baire property), the function f_i is measurable (has the Baire property) too. Since f_i meets all blocking sets, f_i is almost continuous.

Let us put $h_1 = \max(f_1, f_2)$, $h_2 = \max(f_3, f_4)$ and observe that

$f = \min(h_1, h_2)$.

REMARK. Notice that $f = \max(h_1, h_2)$, where $h_1 = \min(\max(f_1,f_2), f_3)$ and $h_2 = \min(\max(f_1, f_3), f_2)$. Indeed, if we define $A_i = \{x: f(x) > f_i(x)\}$ and $B_i = \{x: f(x) < f_i(x)\}$ for $i = 1,2,3$

then $\{x : \max(f_1(x), f_2(x)) \neq f(x)\} = B_1 \cup B_2$ and $\{x : h_1(x) \neq f(x)\} = A_3$.

Similarly, $\{x: h_2(x) \neq f(x)\} = A_2$. Since $f(x) > h_1(x)$ for $x \in A_3$, $f(x) > h_2(x)$ for $x \in A_2$ and $A_2 \cap A_3 = \emptyset$, we obtain $f = \max(h_1, h_2)$.

Of course, for every two functions f,g the lattice generated by the functions f,g is equal to $\{f, g, \max(f,g), \min(f,g)\}$. Because there exist functions which are not maximum or minimum of two functions with Darboux property (see [BCP], Th 3; for example,

$$f(x) = \begin{cases} -1 & \text{for } x = 0, \\ 1 & \text{for } x = 1, \\ 0 & \text{for } x \in (0,1) \end{cases}$$

), then the minimal number $n \in N$ such that every function $f: I \to R$ belongs to the lattice generated by n functions with Darboux property equals three.

COROLLARY. The smallest lattice of functions containing all almost continuous, measurable (with the Baire property) functions is equal to the family of all measurable (with the Baire property) functions.

PROBLEM. Does the lattice generated by the family of all almost continuous, Borel measurable (Borel class α), functions equal the family of all Borel measurable (Borel class α) functions ?

Evidently, if $f,g : I \to I$ have the Darboux property, then the superposition $g \circ f$ has the Darboux property too. Thus the superposition of almost continuous functions has the Darboux property. For a function $f: I \to I$ we say that $f \in D^{**}(I,I)$ iff for every

$y \in I$ and for every subinterval $J \subset I$ the set $f^{-1}(y) \cap J$ has the cardinality c.

PROPOSITION 3. Every function $f \in D^{**}(I,I)$ is a superposition of two almost continuous functions $g,h: I \to U$.

P r o o f. Let $x_0, \ldots, x_\alpha, \ldots$ $(\alpha < c)$ be a well-ordering of all reals $x \in I$ and let $K_0, \ldots, K_\alpha, \ldots$ $(\alpha < c)$ be a well-ordering of all blocking sets in $I \times I$. By induction we choose three sequences of points $(a_\alpha, a'_\alpha) \in K_\alpha$, $(b_\alpha, b'_\alpha) \in K_\alpha$ and $c_\alpha \in I$ $(\alpha < c)$ such that:

(i) $a_\alpha \notin \{a_\beta : \beta < \alpha\}$,

(ii) if $a'_\alpha = a'_\beta$ for some $\beta < \alpha$, then $f(a_\alpha) = f(a_\beta)$,

(iii) if $a'_\alpha = c_\beta$ for some $\beta < \alpha$, then $f(a_\alpha) = f(x_\beta)$,

(iv) if $a'_\alpha = b_\beta$ for some $\beta < \alpha$, then $f(a_\alpha) = b'_\beta$,

(v) $b_\alpha \notin \{b_\beta, a'_\beta, c_\beta : \beta < \alpha\} \cup \{a'_\alpha\}$,

(vi) $c_\alpha \notin \{a'_\beta, b_\beta, c_\beta : \beta < \alpha\} \cup \{a'_\alpha, b_\alpha\}$.

Let us define functions $g,h: I \to I$ as follows.

$$h(x) = \begin{cases} a'_\alpha & \text{for } x = a_\alpha, \quad \alpha < c, \\ c_\alpha & \text{if } x \notin \{a_\alpha : \alpha < c\} \text{ and } x = x_\beta, \end{cases}$$

$$g(x) = \begin{cases} f(a_\alpha) & \text{for } x = a'_\alpha, \quad \alpha < c, \\ b'_\alpha & \text{for } x = b_\alpha, \quad \alpha < c, \\ f(x_\beta) & \text{for } x = c_\beta, \quad \beta < c, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that h and g are almost continuous and $f = g \circ h$.

REMARK. It is easy to find a function $f \in \mathcal{D}^{**}(I,I)$ which does not have the fixed point and consequently, f is not almost continuous

PROBLEM. Is every Darboux function $f: I \to I$ a superposition of (t· almost continuous functions ?

Added in the printing. In the proof of Proposition 3 the assumption that the union of less than the continuum many sets of the first category is of the first category (e.g. if Martin's Axiom or CH holds) is needed.

The construction of (a_α, a'_α) looks as follows. Let J be a non-empty interval which is included in $p_1(K_\alpha)$. For every $x \in J$ we define $y(x) \in I$ such that $(x, y(x)) \in K_\alpha$. For every $\beta < \alpha$ let $A_\beta = \{x \in J: y(x) = a'_\beta\}$, $B_\beta = \{x \in J: y(x) = b_\beta\}$ and $C_\beta = \{x \in J: y(x) = c_\beta\}$. If $D = J \setminus \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta \cup C_\beta \cup \{a_\beta\}) \neq \emptyset$ then we choose $a_\alpha \in D$ and $a'_\alpha = y(a_\alpha)$. Otherwise, e.g. A_γ is of the second category and therefore, $J_1 \subset \overline{A_\gamma}$ for some non-empty subinterval $J_1 \subset J$. Since K_α is closed, $J_1 \times \{a'_\gamma\} \subset K_\alpha$ and we can choose $a_\alpha \in J_1 \setminus \{a_\beta : \beta < \alpha\}$, $a'_\alpha = a'_\gamma$.

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DWIE UWAGI O FUNKCJACH PRAWIE CIĄGŁYCH

Streszczenie

1. Jeżeli $f:R \to R$ jest ciągła i $g:R \to R$ jest prawie ciągła, to funkcje $f+g$, $f \cdot g$, $\max(f,g)$ i $\min(f,g)$ są prawie ciągłe.

2. Każdą funkcję mierzalną (z własnością Baire'a) można przedstawić jako $\min(\max(f_1, f_2), \max(f_3, f_4))$, gdzie funkcje f_1, f_2, f_3, f_4 są prawie ciągłe i mierzalne (z własnością Baire'a).

3. Każdą funkcję $f \in \mathcal{D}^{**}(I,I)$ można przedstawić jako złożenie dwóch funkcji prawie ciągłych $g,h:I \to I$.