ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
Problemy Matematyczne 1988 z.10

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SOME REMARKS ON QUASI-CONTINUOUS REAL FUNCTIONS

Let X be any topological space and let R be the space of real numbers with the natural topology. A function $f:X \to R$ is said to be quasi-continuous at a point $x_0 \in X$ if for every open neighbourhood U of the point x_0 and every open neighbourhood V of the point $f(x_0)$ there is an open set $U' \subset U$, $U' \neq \emptyset$ such that $f(U') \subset V$. A function f is said to be quasi-continous on X if it is quasi-continuous at each point $x \in X$. [2]

A set $A \subset X$ is said to be semi-open if $A \subset \overline{\text{Int } A}$. [1] (by $\bar{A}$ and Int A we denote the closure and the interior of A.)

Lemma 1 [2]

A function $f:X \to R$ is quasi-continuous at $x_0 \in X$ if and only if for every open set V containing $f(x_0)$ the set $f^{-1}(V)$ is semi open.

The oscillation O_f of a real function f defined on topological space is defined by

$$O_f(x_0) = \inf_{U(x_0)} \left\{ \sup_{(x,y) \in U(x_0)} |f(x) - f(y)| \right\},$$

where the infimum is taken under all neighbourhoods $U(x_0)$ of x_0. It is well known that f is continuous at a point x_0 if and only.

if $O_f(x_0) = 0$.

For every real functions f,g by the symbols : $f+g$, $f \cdot g$, $\min\{f,g\}$, $\max\{f,g\}$ we denote the following functions:

$(f+g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \circ g(x)$,

$(\min\{f,g\})(x) = \min\{f(x), g(x)\}$,$(\max\{f,g\})(x) = \max\{f(x), g(x)\}$.

Let us denote:

$C(X,R) = \{f \in R^X$: f is continuous$\}$,

$Q(X,R) = \{f \in R^X$: f is quasicontinuous$\}$,

$A(X,R) = \{f \in R^X$: for every $g \in Q(X,R)$ the sum f+g belongs to $Q(X,R)\}$,

$M(X,R) = \{f \in R^X$: for every $g \in Q(X,R)$ the product $f \cdot g$ belongs to $Q(X,R)\}$,

$E(X,R) = \{f \in R^X$: for every $g \in Q(X,R)$ the maps $\min\{f,g\}$ and $\max\{f,g\}$ belong to $Q(X,R)\}$.

The main aim of this paper is to give characterizations of the classes $A(X,R)$, $E(X,R)$ and $M(R,R)$.

By 0 and 1 we denote the constant functions which assign to any $x \in X$ the number 0 or 1, respectively.

Theorem 1

$A(X,R) = C(X,R)$

Proof : It is obvious that for every continuous function $f:X \to R$ and every quasi-continuous function $g:X \to R$ the sum f+g is quasi-continuous, so $C(X,R) \subset A(X,R)$.

Let f be any function which is not continuous at $x_0 \in X$. If it is not quasi-continuous at x_0 then the sum f+ O is not quasi-continuous at x_0.

Let us suppose that f is quasi-continuous on X. Because $O_f(x_0) > 0$ there is $\alpha > 0$ such that the set $U_1 = f^{-1}((-\infty, f(x_0) - \alpha) \cup (f(x_0) + \alpha, \infty)) \neq \emptyset$ and $x_0 \in \bar{U}_1$. The sets U_1 and $\bar{U}_1$ are semi-open, and the set $V = f^{-1}((f(x_0) - \alpha, f(x_0) + \alpha))$ is nonempty and semi-open. Clearly $\emptyset \neq \operatorname{Int} V \subset X \setminus \bar{U}_1$, so $X \setminus \bar{U}_1$ is nonempty. Put

$$g(x) = \begin{cases} 0 & \text{for } x \in \bar{U}_1 \\ 3\alpha & \text{otherwise.} \end{cases}$$

The function g is quasi-continuous, but because the set $(g+f)^{-1}((f(x_0) - \alpha, f(x_0) + \alpha)) \subset \bar{U}_1 \setminus U_1$, the sum g+f is not quasi-continuous at x_0. This prove that if $f \notin C(X,R)$ then $f \notin A(X,R)$.

Consequently $A(X,R) = C(X,R)$.

Theorem 2

$E(X,R) = C(X,R)$.

Proof: Clearly $C(X,R) \subset E(X,R)$. Let $f: X \to R$ be discontinuous at $x_0 \in X$. If f is not quasi - continuous at x_0, then there is $\alpha > 0$ such that the set $f^{-1}((f(x_0) - \alpha, f(x_0) + \alpha))$ ist not semi-open. Let $g(x) = f(x_0) - 4\alpha$ for every $x \in X$. The function g is continuous. But, since the set $(\max\{f,g\})^{-1}((f(x_0) - \alpha, f(x_0) + \alpha)) = f^{-1}((f(x_0) - \alpha, f(x_0) + \alpha))$ is not semi-open, the function max $\{f,g\}$ is not quasi-continuous at x_0.

Let f be quasi-continuous on X . By discontinuity of f at x_0 there is $\alpha > 0$ such that one of the sets $U_1 = f^{-1}((-\infty, f(x_0)-\alpha))$ $U_2 = f^{-1}((f(x_0)+\alpha, \infty))$ is nonempty and x_0 belongs to its closure. Let $x_0 \in \bar{U}_1$, the function g defined as follows

$$g(x) = \begin{cases} f(x_0) & \text{for } x \in \bar{U}_1 \\ f(x_0) - 2\alpha & \text{otherwise} \end{cases},$$

is quasi-continuous. It is easy to see that the function min $\{f,g\}$ is not quasi-continuous at x_0 .

Now let $x_0 \in U_2$; then in the similar way we may show that there is $g \in Q(X,R)$ such that max $\{f,g\}$ is not quasi-continuous at x_0 .

Finally we have shown that $E(X,R) \subset C(X,R)$ what completes the proof.

It is easy to prove the following lemma .

Lemma 2

If $f: X \to R$ is quasi-continuous and $f(x) \neq 0$ for every $x \in X$ then the function $1/f$ is quasi-continuous.

Theorem 3

A real function f belongs to $M(R,R)$ if and only if the following two conditions are both fulfiled:

1/ f is quasi-continuous ;

2/ if $x_0 \in R$ is a point of discontinuity of f, then $f(x_0) = 0$ and x_0 is the limit of a sequence of points x_n at which f is continuous and $f(x_n) = 0$ for every $n \in N$.

Proof: The "if" implication is clear. For prove the "only if" implication we assume that f is not continuous at a point $x_0 \in R$.

If f is not quasi-continuous then $f \cdot 1$ is not quasi-continuous, so $f \notin M(R,R)$.

Now let f be quasi-continuous, but $f(x_0) \neq 0$. There is $\alpha > 0$ such that the set $U = f^{-1}\left((-\infty, f(x_0) - \alpha) \cup (f(x_0) + \alpha, \infty)\right)$ is semi-open and $x_0 \in \bar{U}$. Let us put

$$g(x) = \begin{cases} 1 & \text{for } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

The function g is quasi-continuous, but the function $f \cdot g$ is not quasi-continuous.

Now we suppose that $f(x_0) = 0$ but there is an open neighbourhood G of x_0 which has not any continuity point x such that $f(x) = 0$. Then, by quasi-continuity of f, the set $B = G \cap f^{-1}(0)$ is nowhere dense in G. The set $\bar{B}$ is a space of the second category in itself /it is a closed subset of the complete metric space R/. For every $x \in \bar{B}$ we have $O_f(x) > 0$; so $\bar{B} = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \{x \in B : O_f(x) \geq 1/n\}$, $n = 1, 2, \ldots$. There is $n_0 \in N$ such that A_{n_0} is of the second category in $\bar{B}$. Because all sets A_n are closed we have $\operatorname{Int}_{\bar{B}} A_{n_0} \neq 0$. We may find an nondegenerated closed interval J having the ends at points of continuity of f such that $\emptyset \neq J \cap B \subset \operatorname{Int}_{\bar{B}} A_{n_0}$. The set $B_1 = J \cap B$ is nowhere dense and compact in J. There is a finite cover $\mathcal{V}_1$ of B_1 such that each member of $\mathcal{V}_1$ is an open interval $(w_1, w_2) \subset J$, we may assume that for every $x \in B_1$ there are at most two members of $\mathcal{V}_1$ containing x and for every $H \in \mathcal{V}_1$ we have $H \cap B_1 \neq \emptyset$. Let $m_1 = \operatorname{card} \mathcal{V}_1$, for every $(w_1^i, w_2^i) \in \mathcal{V}_1$, $i=1,2,\dots, m_1$ there is $t \in (w_1^i, w_2^i)$ such that $|f(t)| > 1/2n_0+1$. Because f is quasi-continuous there is a non-degenerated closed interval $[a^i, b^i] \subset (w_1^i, w_2^i)$ such that for every $z \in [a^i, b^i]$ we have $|f(z)| > 1/2n_0 + 1$ and $[a^i, b^i] \cap B_1 = \emptyset$. Let $D_1 = \bigcup_{i=1}^{m_1} [a^i, b^i]$, then $d(D_1, B_1) = \inf\{|x-y| : x \in D_1, y \in B_1\} > 0$. Let us denote $\mathcal{M}_1 = \{[a^i, b^i]\}_{i=1}^{m_1}$.

Let $k_1 \in N$ be such that $1/k_1 < d(D_1, B_1)$. There is a finite cover $\mathcal{V}_2$ of B_1 such that $\mathcal{V}_2$ has similar properties to $\mathcal{V}_1$ and for every $(p,q) \in \mathcal{V}_2$ we have $|p-q| < 1/k_1$. Let $m_2 = \operatorname{card} \mathcal{V}_2$. Similarly to the above construction we may find a family $\mathcal{M}_2 = \{P_i\}_{i=1}^{m_2}$ of nondegenerated closed intervals having the ends at points of continuity of f such that for every $i=1,2,\dots, m_2$ and every $x \in P_i$ we have $|f(x)| > 1/2n_0 + 1$.
Let $D_2 = \bigcup_{i=1}^{m_2} P_i$, then $d(D_2, B_1) > 0$ and $d(D_2, B_1) < d(D_1, B_1)$.
Let us take $k_2 \in N$ such that $1/k_2 < d(D_2, B_1)$. Continuing this way we may construct a sequence of finite families of nondegenerate closed intervals, denoted by $\{\mathcal{M}_n\}_{n=1}^{\infty}$ such that for any $n \in N$ and any $P \in \mathcal{M}_n$, P has the ends at points of continuity of f and $f(P) \subset (-\infty, -1/2n_0+1) \cup (1/2n_0 + 1, \infty)$ and $\lim_n d(D_n, B_1) = 0$ where $D_n = \bigcup \mathcal{M}_n$. Let us put

$$g(x) = \begin{cases} 1 \text{ for } x \in \bigcup_{i=1}^{\infty} D_i \cup B_1 \cup (R \setminus J) \\ 1/f(x) \quad \text{otherwise.} \end{cases}$$

The function g is quasi-continuous, but we have $(f\cdot g)(x) = 0$ if $x \in B$ and $(f\cdot g)(x) > 1/2\, n_0 + 1$ for $x \in J \setminus B$. This implies that $f\cdot g$ is not quasi-continuous on J and it completes the proof.

The above theorem we have proved for the cas of functions defined on the space of real numbers. Mrs Professor Janina Ewert has noticed that using similar argumentation one may prove this theorem for the case of functions defined on any locally compact metric space. The authors are in debt to Mrs Ewert for her valuable remarks which have enabled to make the proof of Theorem 3 simpler.

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PEWNE UWAGI O FUNKCJACH RZECZYWISTYCH QUASI-CIĄGŁYCH

Streszczenie

Niech (X,T) będzie przestrzenią topologiczną oraz R - zbiorem liczb rzeczywistych. W tym artykule pokazujemy, że jedynie funkcje ciągłe ($X \to R$) mają tę własność, że dodawane do wszystkich funkcji quasi-ciągłych dają w wyniku funkcje quasi-ciągłe oraz że ich maksima i minima z funkcjami quasi-ciągłymi są takie same. Ponadto pokazujemy, że klasa tych funkcji R R, które mnożone przez dowolne funkcje quasi-ciągłe dają iloczyny również quasi-ciągłe jest istotnie większa, niż klasa funkcji ciągłych.