ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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# ON THE FUNCTIONS APPROXIMATELY- AND QUASI-CONTINUOUS WCHICH ARE ALMOST EVERYWHERE DISCONTINUOUS

Denote by R the set of reals numbers.

A function $f : [0,1] \to R$ is said to be quasi-continuous at a point $x_0$, if for every $\varepsilon > 0$ and for every neigborhood U of $x_0$ there exists a nonempty open set $V \subset U$ such that $|f(x) - f(x_0)| < \varepsilon$ for each point $x \in V$.

A function $f : [0,1] \to R$ is said to be quasi-continuous if it is quasi-continuous at each point $x \in [0,1]$ (see [1]).

It is known there exists an approximately continuous function $f : [0,1] \to R$ having the set of discontinuity points of full Lebesque measure (Zahorski [2]). But this Zahorski's function isn't quasi -continuous. I shall prove the following theorem:

Theorem 1. There exists an approximately- and quasi-continuous function having the set of discontinuity points of full Lebesque measure.

Proof. Let $(a_n)_{n=1}^{\infty}$ be sequence of positive reals such that

(1) $$\sum_{n=1}^{\infty} a_n \leqslant 1.$$

Let $C_1$ be a Cantor's set in $[0,1]$ such that $m(C_1) = \frac{1}{2}$ ($m$ denotes the Lebesque measure). Let $F_1 \subset C_1$ be a $F_\sigma$ sets such that $m(C_1 - F_1) = 0$ and every point $x \in F_1$ is a density point of the set $F_1$.

Let $(I_{1k})_{k=1}^{\infty}$ be a sequence of all components of the set $[0,1] - C_1$ and let $(J_{1k})_{k=1}^{\infty}$ be a sequence of closed intervals such that

(2) $$J_{1k} \subset I_{1k} \quad (k = 1,2,\dots);$$

(3) $$\frac{m(J_{1k})}{\operatorname{dist}(J_{1k}, \operatorname{Fr} I_{1k})} < \frac{1}{k} \quad \text{for } k = 1,2,\dots,$$ where $\operatorname{Fr} I_{1k}$

denotes the boundary of the set $I_{1k}$, and $\operatorname{dist}(J_{1k}, \operatorname{Fr} I_{1k}) =$

$$= \inf_{\substack{x \in J_{1k} \\ y \in \operatorname{Fr} I_{1k}}} |x - y| .$$

There is an approximately continuous function $g_1 : [0,1] \to R$ such that $g_1(x) = 0$ for each point $x \in [0,1] - F_1$ and $0 < g_1(x) \leq a_1$ for each $x \in F_1$ ([2]).

Let $h_1 : [0,1] \to [0,a_1]$ be a function such that:

(4) $$h_1(x) = 0 \text{ for each } x \in [0,1] - \bigcup_{k=1}^{\infty} J_{1k} ;$$

(5) $$h_1(J_{1k}) = [0,a_1] \ (k = 1,2,\dots);$$ and

(6) $h_1$ is continuous at each point $x \in [0,1] - C_1$.

The condition (3) implies that the function $h_1$ is approximately continuous. The function

$$f_1 = g_1 + h_1$$

is approximately continuous and by (3), (4), (5) it is quasi--continuous. Let $C_2 \subset [0,1] - C_1$ be a Cantor's set such that $m(C_2) = \frac{1}{4}$. Denote by $(I_{2k})_{k=1}^{\infty}$ a sequence of all components of the set $[0,1] - C_2$. There exists a sequence of closed intervals $(J_{2k})_{k=1}^{\infty}$ such that:

(7) $$J_{2k} \subset I_{2k} \quad (k = 1,2,\dots) \text{ : and}$$

(8) $$\frac{m(J_{2k})}{\operatorname{dist}(J_{2k}, \operatorname{Fr} I_{2k})} < \frac{1}{k} \quad \text{for } k = 1,2,\dots .$$

There exists a function approximately continuous $g_2 : [0,1] \to [0,a_2]$ such that $g_2(x) = 0$ for each $x \in [0,1] - F_2$ and $0 < g_2(x) \leq a_2$ for each $x \in F_2$ ([2]), where $F_2 \subset C_2$ is a $F_\sigma$ set such that $m(C_2 - F_2) = 0$ and every $x \in F_2$ is a density point of $F_2$.

Let $h_2 : [0,1] \to [0,a_2]$ be a function such that:

$$h_2(x) = 0 \quad \text{for each } x \in [0,1] - \bigcup_{k=1}^{\infty} J_{2k} ;$$

(9) $$h_2(J_{2k}) = [0,a_2] \quad (k = 1,2,\dots) \text{; and}$$

(10) $h_2$ is continuous at each point $x \in [0,1] - C_2$.

The condition (8) implies that the function $h_2$ is approximately continuous. Then the function

$$f_2 = g_2 + h_2$$

is approximately continuous and by (9) and (10) it is quasi-continuous.

In generality, we define for n = 1,2,..., the approximately- and quasi-continuous $f_n : [0,1] \to [0,a_n]$, which are continuous at each point $x \in [0,1] - C_n$ and discontinuous at each point $x \in C_n$, where $C_n \subset [0,1] - \bigcup_{k=1}^{n-1} C_k$ is a Cantor's set of measure $\frac{1}{2^n}$.

Let us put

$$(11) \qquad f = \sum_{n=1}^{\infty} f_n .$$

The condition (1) implies the uniform convergence of the serie (11). The uniform convergence of the serie (11) implies the approximately continuity of the function f at each $x \in [0,1]$ and her continuity at each point $x \in [0,1] - \bigcup_{n=1}^{\infty} C_n$.

If $x \in C_{n_0}$, then all function $f_n$ ($n \neq n_0$ and n = 1,2,... ) are continuous at point x and $f_{n_0}$ isn't continuous and is quasi-continuous at this point x. Hence, f is quasi-continuous and isn't continuous at each point $x \in \bigcup_{n=1}^{\infty} C_n$.

But

$$m\left(\bigcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} m(C_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

the set of discontinuity points of f is of full Lebesgue measure.

We consider the space of all bounded, approximately- and quasi-continuous functions $f,g : [0,1] \to R$ with Tchebyschev metric

$$\varrho(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Theorem 2. Let $\varepsilon$ be a positive number and let $f : [0,1] \to R$ be a bounded approximately- and quasi-continuous function. There exists a bounded approximately- and quasi-continuous function $g : [0,1] \to R$ having the set of discontinuity points of full Lebesque measure and such that $\varrho(f,g) \leq \varepsilon$.

Proof. It the set $D(f)$ of all discontinuity points of $f$ is of full Lebesque measure, then $g = f$.
If $m(D(f)) < 1$, then we can show of the proof of the theorem 1 that there exists au approximately- and quasi-continuous function $h : [0,1] \to [0,\varepsilon]$ which is discontinuous at each point $x$ of a set $E \subset C(f)$ ($C(f)$ denotes the set of all continuity points of $f$) such that $m(C(f) - E) = 0$ and continuous at each point $x \in [0,1] - E$. Hence the function $g = f + h$ satisfies all required conditions.

Theorem 3. Let $\varepsilon$ be a positive number and let $f : [0,1] \to R$ be a bounded approximately- and quasi-continuous function. There exists a bounded approximately- and quasi-continuous function $g : [0,1] \to R$ having the set of continuity points of positive Lebesgue measure and such that $\varrho(f,g) \leq \varepsilon$.

Proof. If the set $C(f)$ of continuity points of $f$ is of positive measure, then $g = f$. If $m(C(f)) = 0$, let

$$A_{\frac{\varepsilon}{2}}(f) = \left\{ x \in [0,1] \; ; \text{ the oscilation of } f \text{ at point } x \text{ is } < \frac{\varepsilon}{2} \right\}.$$

The set $A_{\frac{\varepsilon}{2}}(f)$ is open and dense. Let $I = [a,b] \subset A_{\frac{\varepsilon}{2}}(f)$ $\left(a < b \text{ and } \operatorname{osc}_I f < \frac{\varepsilon}{2}\right)$ a closed interval such that $f(a) = f(b)$. There exists such intervalle, because monotone approximately continuous function is continuous. Define

$$h(x) = \begin{cases} -f(x) + f(a) & \text{if } x \in I \\ 0 & \text{if } x \in [0,1] - I. \end{cases}$$

Then the function $g = f + h$ is approximately- and quasi-continuous, $g/_I \equiv f(a)$ and $|f - g| \leqslant \varepsilon$. Because $C(g) \supset \operatorname{Int} I$, we have $m\left(C(h)\right) > 0$.

This complete the proof.

Remarque. Let

$$AQ = \left\{ f : [0,1] \to R \; ; \; f \text{ is bounded, approximately- and quasi-continuous} \right\},$$

$$AQ_0 = \left\{ f \in AQ \; ; \; m\left(C(f)\right) = 0 \right\} \text{ and}$$

$$AQ_1 = \left\{ f \in AQ \; ; \; m\left(C(f)\right) > 0 \right\}.$$

From the theorems 2 and 3 we see that the sets $AQ_0$ and $AQ_1$ are dense both in AQ.

Corollary. The set $AQ_1$ is a residual $G_\delta$ set in AQ.

Proof. Because $AQ_1 = \bigcap_{n=1}^{\infty} \left\{ f \in AQ \; ; \; m\left(C(f)\right) \leqslant \frac{1}{4} \right\}$ and all the sets $\left\{ f \in AQ \; ; \; m\left(C(f)\right) \leqslant \frac{1}{n} \right\}$ are open ([3], Lemma 1), so $AQ_1$ is a $G_\delta$ set in AQ. Every a dense $G_\delta$ set is residual.

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O FUNKCJACH APROKSYMATYWNIE - ORAZ QUASI-CIĄGŁYCH, KTÓRE SĄ PRAWIE WSZĘDZIE NIECIĄGŁE

Streszczenie

W tym artykule pokazuję, że w przestrzeni funkcji ograniczonych $f : [0,1] \to R$ aproksymatywnie- i quasi-ciągłych z metryką Czebyszewa zarówno zbiór funkcji prawie wszędzie nieciągłych, jak i jego dopełnienie są gęste.