ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ
W BYDGOSZCZY
Problemy Matematyczne 1982 z.4

Andrzej Nowicki
UMK Tourń

Ryszard Żuchowski
WSP Bydgoszcz

SPECTRAL SPACES AND RADICALS IN SYSTEMS OF IDEALS

1. Introduction. In this paper, we study sets of ideals of a commutative ring R with identity, which are closed undor any intersections and containing R. If E satisfies these conditions, then by E^* we denote the set of all E-prime ideals i.e.ideals P in E, for which the condition $AB \subset P$, where A, B belong to E, implies $A \subset P$ or $B \subset P$. We introduce in E^* a topology and we prove, that if E satisfies an additional condition, then there exists a ring S, such that the spaces Spec S and E^* are homeomorphic.

Simultaneously, we get some properties so-called r_E -radical ideals i.e. ideals of E, which are intersections of E -prime idals. Further, we examine systems of ideals in the sense [8], [9] , and we give an application of this theory for a description of some distinguish subsets of these systems and for a description of radicals, which are connected with these subsets.

These subsets: A(R, M), B(R, M), C(R, M) were introduced in [8], [9] . We shall prove, that if we give an additional assumption, every of these class is a spectral space.

Throughout this paper, all rings are commutative with unity. Let R be a ring. By I(R) we denote the set of all ideals in R. If T is any subset of R, then by r(T) we denote a radical of T i.e. an intersection of all prime ideals containing T.

An ideal A of R is radical iff $r(A) = A$. If A is an ideal of R, and $x \in R$, then by A_x we denote the ideal $\bigcup_{n=0}^{\infty}(A:x^n)$ where

$A:B = \left\{ r \in R; \bigwedge_{b \in B} r \cdot b \in A \right\}$. It is easy to verify that $r(A_x) = (r(A) : x)$. Let (R, M) be a system of ideals in the sense [7], [8], [9]. Ideals of M will be called M-ideals.

An M-ideal P is primitive, if there exists a multiplicatively closed subset S of R such that $P \cap S = \emptyset$, and P is maximal among M-ideals of R, disjoint from S. The set of primitive M-ideals will be denoted by $B(R, M)$

An M-ideal P is called M-prime, if for M-ideals A, B the condition $A.B \subset P$, implies $A \subset P$ or $B \subset P$. The set of all M-prime ideals of (R, M) will be denoted by $C(R, M)$. Moreover, by $A(R, M)$ will be denoted the set of all prime ideals of R, which belong to M.

By some modifications of the proof of the theorem at [5], for differential rings, one can prove, that $A(R, M) \subset B(R, M) \subset C(R, M)$.

If T is any subset of R, then by $[T]$ we denote the smallest M-ideal containing T. If A is an ideal of R, then by $A_{\#}$ we denote the greatest M-ideal contained in A.

2. E-prime ideals. Let R be a ring, $I(R)$ - the set of all ideals of R, and let E be a subset of $I(R)$ such that $R \in E$. Ideals of E are called E-ideals. An E-ideal $P \neq R$ is called E-prime, if for E-ideals A, B, the condition $A.B \subset P$, implies $A \subset P$ or $B \subset P$.

The set of all E-prime ideals will be denoted by E^*. If T is any subset of R, then the intersection of all E-prime ideals, which contains T, will be denoted by $r_E(T)$, and will be called an E-radical of T. (If there is no E-prime ideals containing T, then we set: $r_E(T) = R$.

An E-ideal A will be called E-radical iff $r_E(A) = A$.

PROPOSITION 2.1. If T, S are subsets of R, A, B are E-ideals and $P \in E^*$ then

(1) $T \subset r_E(T)$

(2) $T \subset P$ if and only if $r_E(T) \subset P$

(3) If $T \subset S$, then $r_E(T) \subset r_E(S)$

(4) $r_E(r_E(T)) = r_E(T)$

(5) $r_E(P) = P$

(6) $r_E(A + B) = r_E(r_E(A) + r_E(B))$

(7) $r_E(AB) = r_E(A \cap B) = r_E(A) \cap r_E(B)$

PROOF. . (1), (2), (3), (5) are obvious.

(4). Since (2), we have $\{P \in E^*;\ P \supset T\} = \{P \in E^*;\ P \supset r_E(T)\}$, so $r_E(r_E(T)) = \bigcap \{P \in E^*;\ P \supset r_E(T)\} = \bigcap \{P \in E^*;\ P \supset T\} = r_E(T)$.

(6). Since (3), we have the inclusion $\subset$. Now, if $P \supset A + B$, then $P \supset A$ and $P \supset B$, thus $P \supset r_E(A)$ and $P \supset r_E(B)$. Finally, we get $P \supset r_E(A) + r_E(B)$.

(7). By (3), we get $r_E(AB) \subset r_E(A \cap B) \subset r_E(A) \cap r_E(B)$, it suffices to show, that $r_E(AB) = r_E(A) \cap r_E(B)$. Let $I = \{P \in E^*;\ P \supset AB\}$, $J = \{P \in E^*;\ P \supset A\}$, $K = \{P \in E^*;\ P \supset B\}$. Since $I = J \cup K$, we have $r_E(AB) = \bigcap_{P \in I} P = \bigcap_{P \in J \cup K} P =$

$= \bigcap_{P \in J} P \cap \bigcap_{P \in K} P = r_E(A) \cap r_E(B)$.

If T is any subset of R, then by $V_E(T)$ we shall denote the set of all E-prime ideals containing T. The condition (2) of Proposition 2.1 implies $V_E(T) = V_E((T)) = V_E(r_E(T))$.

PROPOSITION 2.2. If E is a set of ideals in R, $R \in E$ and E is closed under any intersections, then

(1) $V_E(0) = E$, $V_E(1) = \emptyset$

(2) If $\{T_i\}_{i \in I}$ is a family of subsets of R then

$$\bigcap_{i \in I} V_E(T_i) = V_E\left(\bigcup_{i \in I} T_i\right)$$

(3) If T_1, T_2 are subsets of R, then $V_E(T_1) \cup V_E(T_2) = V_E(r_E(T_1)\, r_E(T_2))$

Let $E \subset I(R)$ be a set closed under any intersections and containing R. By the above Proposition, E^* is a topological space with the closed sets $V_E(T)$. The open sets in E^* have the form

$D_E(T) = \{ P \in E^*, P \not\supset T \}$, where T is a subset of R.

If $Y \subset E^*$, we denote $J_E(Y) = \bigcap \{P, P \in Y\}$, if $Y \neq \emptyset$ and $J_E(Y) = R$, if $Y = \emptyset$. It is clear, that $J_E(Y) \in E$.

PROPOSITION 2.3. (a) If $T \subset R$, then $J_E V_E(T) = r_E(T)$

(b) If $Y \subset E^*$, then $V_E J_E(Y) = \bar{Y}$, where $\bar{Y}$ is the closure of Y in E.

PROOF. (a) $J_E V_E(T) = \bigcap \{ P, P \in V_E(T)\} = \bigcap \{P \in E^*, P \supset T\} = r_E(T)$.

(b) Since $Y \subset V_E(J_E(Y))$, we have $\bar{Y} \subset V_E J_E(Y)$.

Conversely, if $P \in E^*$ and $P \supset J_E(Y)$, and we assume that $P \notin \bar{Y}$, then there exists and E -ideal A such, that $D_E(A) \cap Y = \emptyset$, $P \in D_E(A)$. Then we have $Y \subset V_E(A)$, hence $P \supset J_E(Y) \supset A$ and $P \in V_E(A)$. It contradicts with the fact that $P \in D_E(A)$.

COROLLARY 2.4. If $P \in E^*$, then $V_E(P) = \overline{\{P\}}$.

COROLLARY 2.5. There is a bijection between the set of closed subsets of E^* and the set of E-radical E-ideals in R. The mappings V_E, J_E are order-reversing bijections.

PROPOSITION 2.6. If $P \in E^*$, then $V_E(P)$ is a non -empty irreducible closed subset of E. Every non-empty irreducible closed subset of E has the form $V_E(P)$, where $P \in E^*$.

PROOF. Let $P \in E^*$, and let $V_E(P) \subset V_E(r_E(A)) \cup V_E(r_E(B))$. Then $P \supset r_E(A)\, r_E(B)$, whence $P \supset r_E(A)$ or $P \supset r_E(B)$ so we have $V_E(P) \subset V_E(r_E(A))$ or $V_E(P) \subset V_E(r_E(B))$. Suppose now, that V is a non-empty closed irreducible subset in E^*, and $V = V_E(Q)$, where $Q \in E$ and $r_E(Q) = Q$. We show, that $Q \in E^*$. If $Q \supset AB$, where $A, B \in E$, then $V_E(Q) \subset V_E(AB) = V_E(A) \cup V_E(B)$. Since

$V(Q)$ is irreducible, $V_E(Q) \subset V_E(A)$ or $V_E(Q) \subset V_E(B)$. Hence we have $Q \supset A$ or $Q \supset B$.

A set E is said to be r_E-Noetherian, if E satisfies the ascending chain condition on E-radical ideals.
The following lemma is obvious.

LEMMA 2.7. Let E be an r_E-Noetherian set. If T is a subset of R, then there exists a finite set T_O, such that $T_O \subset T$ and $r_E(T) = r_E(T_O)$.

PROPOSITION 2.8. If E is an r_E-Noetherian, then every open set in E^* is quasi-compact.

PROOF. First we prove, that if $x \in R$, then $D_E(x)$ is an open quasi-compact set in E. Because for $T \subset R$, we have $D_E(T) = \bigcup_{t \in T} D_E(t)$ it suffices to show, that if $D_E(x) = \bigcup_{i \in I} D_E(x_i)$, where $x_i \in R$ for each $i \in I$, then $D_E(x) = D_E(x_{i_1}) \cup \ldots \cup D_E(x_{i_n})$
By the Lemma 2.7., we have

$$D_E(x) = \bigcup_{i \in I} D_E(x_i) = D_E(\{x_i \,;\, i \in I\}) = D_E(r_E(\{x_i \,;\, i \in I\})) =$$

$$= D_E(r_E(\{x_{i_1}, \ldots, x_{i_n}\})) = D_E(\{x_{i_1}, \ldots, x_{i_n}\}) = D_E(x_{i_1}) \cup \ldots \cup D_E(x_{i_n}).$$

Now, if $D_E(A)$ is any open set in E, then

$$D_E(A) = D_E(r_E(A)) = D_E(r_E(\{a_1, \ldots, a_n\})) = D_E(\{a_1, \ldots, a_n\}) =$$

$$= D_E(a_1) \cup \ldots \cup D_E(a_n), \text{ where } a_1, a_2, \ldots, a_n \in A .$$

$D_E(A)$, as a finite union of quasi-compact open sets, is quasi-compact.

COROLLARY 2.9. If E is an r_E-Noetherian set, then E^* is a quasi-compact space.

THEOREM 2.10. Let E be a set of ideals of R, closed under any intersections and $R \in E$. If E is an r_E- Noetherian set, then there exists a ring S such, that the spaces Spec S and E^* are homeomorphic.

PROOF. Applying the results of [2], it suffices to show, that E^* has the following properties

a) E^* is T_0 - space.

b) E^* is quasi-compact,

c) The quasi-compact open subsets of E^* are closed under finite intersections.

d) The quasi-compact open sets in E^* form an open basis.

e) Every non-empty irreducible closed subset has a generic point.

The property a) is obvious and is satisfied by any set E of ideals of R, closed under intersections and which possesses R.

b) - Corollary 2.9.

c), d) - Proposition 2.8.

e) - Proposition 2.6.

Now, we give some applications of the above theory for the sets: A(R, M), B(R, M), C(R, M), where (R, M) is a system of ideals in the sens [8], [9] .

<u>3. The space A(R, M).</u> Let (R, M) be a system of ideals, and let A(R, M) be the set of all prime ideals, which belong to M. Moreover, let $E = \{ G \in M;\ r(G) = G \}$. The set E is closed under any intersections and $R \in E$, so E satisfies the conditions of the Part 2. We shall prove, that $E^* = A(R, M)$.

If T is any subset of R, then by $r_A(T)$ we denote the smallest radical M-ideal in (R, M) containing T.

LEMMA 3.1. If $G \in E$, then $(G:T) \in E$, for every subset T of R.

PROOF. Let $x \in R$. Then $G_x = (G:x)$ and $r(G:x) = r(G_x) = (r(G) : x) = (G:x)$, so $(G:x) \in E$. Now, if T is any subset of R, then $(G:T) = \bigcap_{x \in T} (G:x)$, and hence we have $(G:T) \in E$.

PROPOSITION 3.2. If S, T are subsets of R, then $r_A(S) \cap r_A(T) = r_A(ST)$.

PROOF. (this method is similar to the method of R.M. Cohn [1]). First, we prove that $r_A(S)r_A(T) \subset r_A(ST)$. Since $T \subset (r_A(ST) : S)$ and $(r_A(ST) : S) \in E$ (Lemma 3,1.), we have $r_A(T) \subset (r_A(ST) : S)$, that means $S \subset (r_A(ST) : r_A(T))$ and hence $r_A(S) \subset (r_A(ST) : r_A(T))$ ie. $r_A(T)r_A(S) \subset r_A(ST)$
Further, we have $(r_A(S) \cap r_A(T))^2 = (r_A(S) \cap r_A(T))(r_A(S) \cap r_A(T)) \subset r_A(S)r_A(T) \subset r_A(ST)$, that implies $r_A(S) \cap r_A(T) \subset r_A(ST)$.

The inverse inclusion is obvious.

THEOREM 3.3. If (R,M) is a system and $E = \{G \in M;\ r(G) = G\}$, then $E^* = A(R,M)$.

PROOF. It suffces to prove, that every ideal in E^* is prime, Let $P \in E^*$, $x,y \in P$, $xy \in P$. Then $r_A(x)r_A(y) \subset r_A(xy) \subset r_A(P) = P$, and hence $r_A(x) \subset P$ or $r_A(y) \subset P$, that means $x \in P$ or $y \in P$.

THEOREM 3.4. If (R,M) is a system of ideals and $E = \{G \in M, r(G) = G\}$ then $r_A(T) = r_E(T)$, for every subset T of R.

PROOF. It suffices to verify (by Theorem 3.3.) that $r_A(T) = \bigcap \{P \in A(R,M);\ P \supset T\}$ Let $G = r_A(T)$. If $G = R$, then the thesis is trivial. Suppose now that $G \not\subset R$ and $x \quad R \setminus G$. Consider an inducitive family $\sum_x = \{H \in E ;\ G \subset H,\ x \notin H\}$. Let P be a maximal element in $\sum_x$. We prove, that $P \in A(R,M)$. Therefore it suffices to demonstrate, that P is a prime ideal in R. Suppose, that $uv \in P$, $u \notin P$, $v \notin P$.
Then $P \not\subset r_A(P,u)$, $P \not\subset r_A(P,v)$, so

$s \in r_A(P,u) \cap r_A(P,v) = r_A((P,u)(P,v)) \subset r_A(P) = P$, it gives a contradiction with $x \notin P$. So P is a prime ideal, and we proved that for every $x \in R \setminus G$ there exists a prime ideal $P_x \in A(R,M)$ such that $x \notin P_x$, $G \subset P_x$. Finally, we have $G = \bigcap \{P_x ;\ x \quad R \setminus G\}$.
By the Theorems 2.10., 3.3 we have

COROLLARY 3.5. If (R,M) is a system which satisfies the ascending chain condition on r_A - radical M-ideals, than there exists a ring S such that spaces $A(R, M)$ and Spec S are homeomorphic.

Other proofs of Corollary 3.5, we can find in the paper [8] (for special systems) and [6].

Now, we give another description of the ideal $r_A(T)$. If T is any subset of R, then we denote:

$$\{T\}_0 = T$$
$$\{T\}_{n+1} = r([\{T\}_n]), \text{ for } n \geqslant 0.$$

THEOREM 3.6. Let (R, M) be a system. If T is a subset of R, then $r_A(T) = \bigcup_{n=0} \{T\}_n$.

PROOF. Let $H = \bigcup_{n=0} \{T\}_n$. It is obvious, that $r(H) = H$. If $x \in H$, then $x \in \{T\}_n \subset [\{T\}_n]$, for some n, therefore $[x] \subset$ $\subset [\{T\}_n] \subset \{T\}_{n+1} \subset H$. By induction, it is easy to prove, that $\{T\}_n \subset$ $\subset r_A(T)$, for every $n \in N$, hence $H \subset r_A(T)$.

THEOREM 3.7. If (R, M) is a system, then for any subsets S, T of R, and for every $n, m \in N$ holds: $\{S\}_n \{T\}_m \subset \{ST\}_{n+m}$

PROOF. We prove this theorem in several parts.

a) First, we prove that, if $x \in R$, then $x[T] \subset r([x])$.
Since $T \subset ([xT] : x) \subset \bigcup_{n=0}^{\infty} ([xT] : x^n) \in M$, so we have $[T] \subset$ $\subset \bigcup_{n=0}^{\infty} ([xT] : x^n) \subset r(\bigcup_{n=0}^{\infty} ([xT] : x^n)) = r([xT] : x)$.

b) We demonstrate, that $r(S)r([T]) \subset r([ST])$. By the part a), we have $S[T] \subset r([ST])$. Hence, it follows that $r(S)r([T]) \subset r(S) \cap$ $\cap r([T]) = r(S[T]) \subset r(r([ST])) = r([ST])$.

c) By induction, we shall prove, that $(S)\{T\}_n \subset \{ST\}_n$

For $n = 0$, the inclusion is trivial. Suppose , that this inclusion is satisfied for some n. Then $(S)\{T\}_{n+1} = (S)\, r([\{T\}_n]) \subset$ $\subset r(S)\, r([\{T\}_n]) \subset r([S\{T\}_n]) \subset r([\{ST\}_n]) = \{ST\}_{n+1}$

d) We set one m. By induction with respect to n, we prove that $\{S\}_n \{T\}_m \subset \{ST\}_{n+m}$. For $n = 0$, the inclusion follows from c).

Further, we have $\{S\}_{n+1}\ \{T\}_m = r([\{S\}_n])\ \{T\}_m \subset r([\{S\}_n\ \{T\}_m]) \subset$
$\subset r([\{ST\}_{n+m}]) = \{ST\}_{n+m+1}$

4. The space $C(R, M)$. Let (R, M) be a system of ideals and let $C(R, M)$ be the set of all M-prime ideals in M. If T is any subset of R, then by $r_C(T)$, we denote the intersection of all prime ideals containing T. Let E = M. the set E is closed under any intersections and contains R, so it satisfies the assumption of the part 2. Using earlier notations, we have $E = C(R, M)$, $r_E(T) = r_C(T)$.

By the Theorem 2.10. we have

COROLLARY 4.1. If (R, M) is a system satisfying the ascending chain condition on r_C-radical M-ideals, then there exists a ring S such that the spaces $C(R, M)$ and Spec S are homeomorphic.

Now, we give some properties r_C-radical ideals, that means such M-ideals G, for which $G = r_C(G)$. We say, that an M-ideal is M-irreducible, if it isn't an intersection of two M-ideals, which properly contains its.

PROPOSITION 4.2. Every M-irreducible r_C-radical ideal is M-prime.

PROOF. Let P be an M-irreducible r_C-radical ideal and let A, B be M-ideals such that $AB \subset P$. Then $(A + P)\ (B + P) \subset P$ and we have $P = r_C(A + P) \cap r_C(B + P)$. Indeed, $P = r_C(P) \supset r_C$ $\supset r_C((A + P)\ (B + P)) = r_C(A + P) \cap r_C(B + P) \supset r_C(P) = P$. Because P is M-irreducible, so $P = r_C(A + P)$ or $P = r_C(B + P)$. Finally $A \subset P$ or $B \subset P$.

PROPOSITION 4.3. If (R, M) is a system satisfying the ascending chain condition on r_C-radical ideals, then every r_C-radical ideal is a finite intersection of M-ideals.

PROOF. Suppose, that the set Σ of all r_C-radical ideals which are not finite intersections of M-ideals, is non-empty. Let P be a maximal element of $\sum$. By the Proposition 4.2, P is not M-irreducible. Therefore, there exist M-ideals A, B such that $P \subsetneq A$, $P \subsetneq B$ and $A \cap B = P$. Since $r_C(A)$, $r_C(B)$ don't belong to $\sum$, then

$$r_C(A) = A_1 \cap A_2 \cap \ldots \cap A_k$$

$$r_C(B) = B_1 \cap B_2 \cap \ldots \cap B_l,$$

where $A_1, \ldots, A_k, B_1, \ldots, B_l \in C(R, M)$.

Now, we have

$$P = r_C(P) = r_C(A \cap B) = r_C(A) \cap r_C(B) = A_1 \cap \ldots \cap A_k \cap$$
$$\cap B_1 \cap \ldots \cap B_l$$

this contradicts with the fact that $P \in \sum$.

COROLLARY 4.4. If (R, M) is a system of ideals satisfying the ascending chain condition on r_C-radical ideals then for every M-ideal A there exist only a finite set of minimal M-prime ideals which contain A

<u>5. The space B(R, M).</u> Let (R, M) be a system of ideals and let B(R, M) be the set of all primitive M-ideals.

PROPOSITION 5.1. The following conditions are equivalent

(1) $A \in B(R, M)$

(2) A is a prime ideal and $A = r(A)_{\#}$

(3) r(A) is a prime ideal and $A = r(A)_{\#}$

(4) There exists a prime ideal P in R, such that $A = P_{\#}$.

PROOF. Is similar to the proof of the analogous theorem for differential rings ([4] Prop. 2.2).

If T is any subset of R, then by $r_B(T)$ we denote the intersection of all primitive M-ideals containing T.

THEOREM 5.2. If T is a subset of R, then $r_B(T) = r([T])_{\#}$.

PROOF. If P is a prime ideal containing T, then by Proposition 5.1., $P_{\#}$ is primitive and $P_{\#} \supset [T] \supset T$. Then $r_B(T) \subset r([T])$, and whence we have $r_B(T) \subset r([T])_{\#}$.

If Q is a primitive ideal containing T, then by Proposition 5.2, we have that $r(Q)$ is a prime ideal containing $r([T])$ and $r(Q)_{\#} = Q$. Therefore $Q = r(Q)_{\#} \supset r([T])_{\#}$, and finally $r_B(T) \supset r([T])_{\#}$.

COROLLARY 5.3. If G and H are M-ideals, then $r_B(GH) = r_B(G \cap H) = r_B(G) \cap r_B(H)$.

PROOF. $r_B(GH) = r([G \cdot H])_{\#} = r(GH)_{\#} = (G \cap H)_{\#} = r([G \cap H])_{\#} = r_B(G \cap H)$. $r_B(GH) = r([GH])_{\#} = r(GH)_{\#} = (r(G) \cap r(H))_{\#} = r(G)_{\#} \cap r(H)_{\#} = r([G])_{\#} \cap r([H])_{\#} = r_B(G) \cap r_B(H)$.

An M-ideal G is called r_B-radical, if $r_B(G) = G$.

PROPOSITION 5.4. If (R, M) is a system satisfying the ascending chain condition on r_B- radical ideals , then there exists a ring S such that the spaces B(R,M) and Spec S homeomorphic. The proof is similar to the analogous proof of the theorem for differential rings.

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PRZESTRZENIE SPEKTRALNE ORAZ RADYKAŁY W SYSTEMACH IDEAŁÓW

STRESZCZENIE

W niniejszej pracy zajmujemy się rodzinami ideałów pierścienia przemiennego R z jedynką, zamkniętymi ze względu na dowolne przekroje i zawierającymi R. Jeżeli E jest taką rodziną, to przez E^* oznaczmy zbiór wszystkich ideałów E-pierwszych. Wprowadzamy w E^* topologię i udowadniamy, że przy pewnym założeniu dodatkowym istnieje pierścień S taki, że przestrzenie Spec S i E^* są homeomorficzne.

Jednocześnie, otrzymujemy szereg własności tzw. ideałów r_E-radykalnych, to znaczy takich ideałów z E, które są przekrojami ideałów E-pierwszych. W dalszym ciągu koncentrujemy się na systemach ideałów w sensie [8], [9] i podajemy zastosowanie powyższej teorii do opisu pewnych wyróżnionych podzbiorów takich systemów oraz do opisu radykałów związanych z tymi podzbiorami.